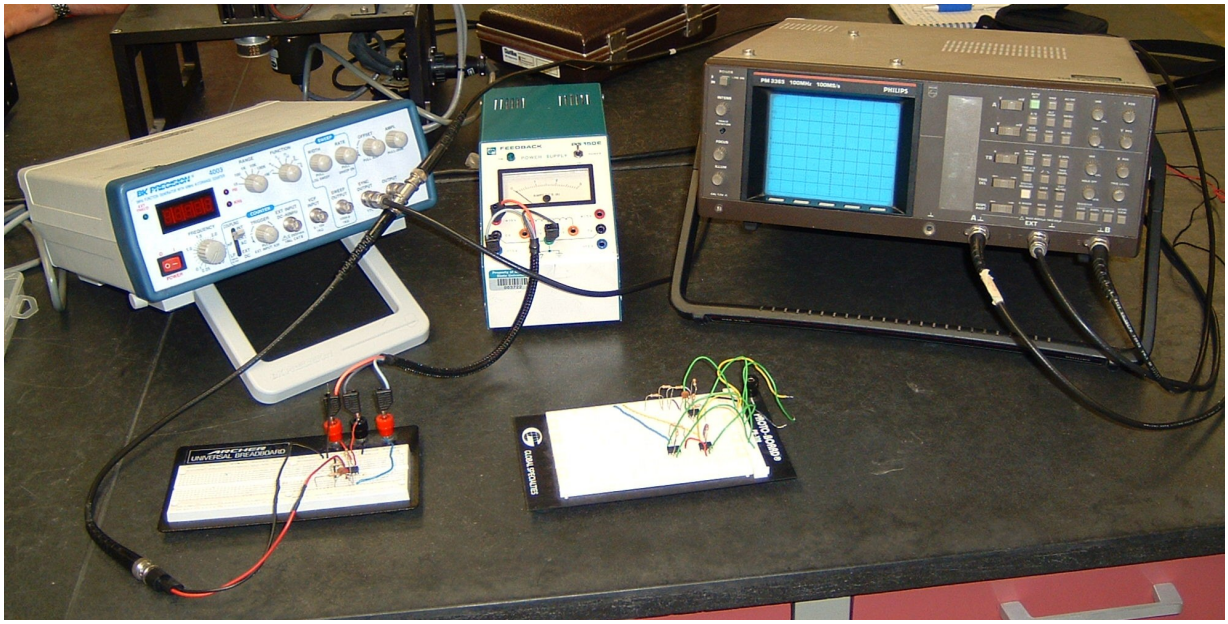
ME 451 : Control Systems Laboratory

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ME451 Laboratory

Analysis and Design of a First Order Plant: Operational Amplifier Circuit



ME451 Laboratory Manual Pages, Last Revised: September 1, 2009
Send comments to: Dr. Clark Radcliffe

Reference

C.L. Phillips and R.D. Harbor, *Feedback Control Systems*, Prentice Hall, 4th Ed.
Section 2.2, pp. 15-22: Electrical Circuits
Section 4.1, pp. 116-121; Time Response of First Order Systems
Section 4.4, pp. 129-132; Frequency Response of First Order Systems
Appendix B, pp. 635-650:Laplace Transform
(Particularly the "Final Value Theorem…)

1. Objective

The response of a linear system to a sinusoidal input is useful for predicting its response to arbitrary periodic inputs and, more importantly, for compensator design. For first-order systems, the sinusoidal response depends on both the DC gain, K, and the time constant, τ. Both, K and τ are functions of system parameters. The objective of this experiment is to investigate the effect of system parameters on system response to a sinusoidal input. We will experiment with an electrical circuit comprised of an operational amplifier, resistors, and a capacitor, that behaves as a first-order system. We will identify the system parameters that affect the DC gain and time constant, obtain the transfer function of the system, and examine its relation to the sinusoidal response. We will then change the system parameters and predict and verify the system response to demonstrate the quality of our model.

2. Background

2.1. Sinusoidal Signals:

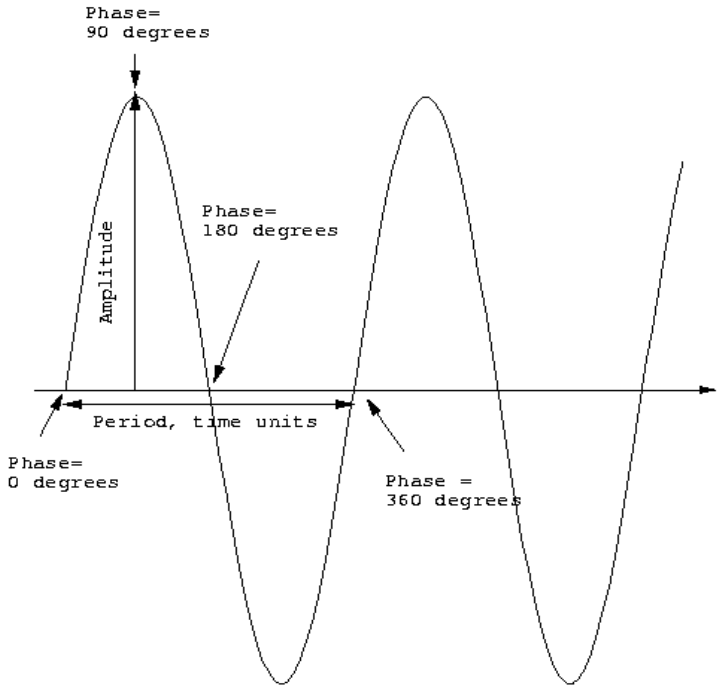


Figure 1. A Sinusoidal Signal.

A sine wave: is characterized by 3 parameters:

- amplitude A_c
- frequency ω (radians/sec), and
- phase ϕ (radians)

The phase ϕ (radians) of a sine wave is a relative quantity; since the sine function can take any argument and has no absolute starting point. The conventional starting point for a sine wave is taken to be the positive, or rising, zero crossing. When we discuss a single sine function, without specifically stating otherwise, it is typically implied that the phase is measured against this standard. A sine wave $y(t)$ varying with time *t* is written as:

$$y(t) = A_c \sin(\omega t + \phi) \tag{1}$$

Here, the amplitude, A_c, has whatever units (feet, volts, psi, etc.) the quantity $y(t)$ represents physically. Phase angle ϕ has angular units of radians while ω has a unit of radians per time unit, because the quantity ωt must have units of angle. The angular frequency ω (rad/sec) is **2π** times the circular frequency ***f*** (cycles/sec = Hertz),

$$\omega = 2\pi f \tag{2}$$

The reciprocal of frequency f (cycles/sec) is the period T (sec/cycle), i.e. the time occupied by a complete cycle of the sine wave.

$$T = \frac{1}{f}$$

2.2. Sinusoidal Response:

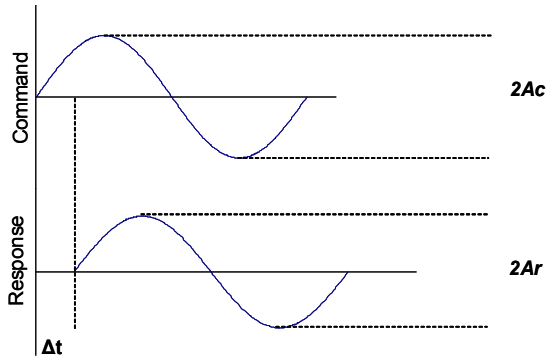


Figure 2. Sinusoidal Command versus Sinusoidal Response.

Consider a linear process with a sinusoidal input, whose output is observed. If input and output are examined together (Fig. 2), then the command input can be compared with the response output. For any linear system,

- The output will be a sinusoid of the same frequency as the input.
- The ratio of the output amplitude to that of the input amplitude (often called process gain) will in general vary with the frequency of the sine wave input.
- The difference in phase between the input and output sine waves will also depend on the frequency.

We call the ratio of the output to input the process **Gain**

$$G(\omega) = \frac{y_R(t)}{y_C(t)} = \frac{A_R \sin(\omega t + \phi_R)}{A_C \sin(\omega t + \phi_C)} \tag{3}$$

where A_R and A_C are the amplitudes of the output and input signals respectively, ϕ_R and ϕ_C are the phases of the output and input signals with respect to the standard sine wave, ω is the frequency of both sine waves. The **gain amplitude** is $|G(\omega)| = A_R / A_C$, and the difference between input and output phases, $(\phi_R - \phi_C)$, is the **phase angle.** The phase can be found by measuring the time, Δt , between an event (peak or zero crossing) on the input signal and the corresponding event on the output signal. This can be done with cursors on an oscilloscope. Once Δt has been measured, equation (4) can be used to calculate the phase angle.

$$\angle G(\omega) = (\phi_R - \phi_C) = \frac{2\pi}{T}\Delta t \text{ (radians)} = \frac{360}{T}\Delta t \text{ (degrees)} \tag{4}$$

2.3. Frequency Response of a First-order systems:

The standard form of transfer function of a first-order system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1} \tag{5}$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the output $y(t)$ and input $u(t)$, respectively, K is the DC gain, and τ is the time constant. For a sinusoidal input

$$u(t) = A\sin(\omega t), \quad U(s) = \frac{A\omega}{s^2 + \omega^2} \tag{6}$$

and the response of the system, in Laplace domain, can be written as

$$Y(s) = \frac{KA\omega}{(s^2 + \omega^2)(\tau s + 1)} \tag{7}$$

Applying partial fraction expansion, equation (7) can also be written as

$$Y(s) = B\left(\frac{s}{s^2 + \omega^2}\right) + C\left(\frac{\omega}{s^2 + \omega^2}\right) + D\left(\frac{1}{s + 1/\tau}\right) \tag{8}$$

where: $B = \frac{-KA\omega\tau}{(\omega\tau)^2 + 1}$, $C = \frac{KA}{(\omega\tau)^2 + 1}$, and $D = \frac{KA\omega\tau^2}{(\omega\tau)^2 + 1}$.

The time response $y(t)$ is found by looking up the inverse Laplace transform of (8) to find

$$y(t) = \frac{-K A\omega\tau}{(\omega\tau)^2+1}\cos(\omega t) + \frac{KA}{(\omega\tau)^2+1}\sin(\omega t) + \frac{KA\omega\tau^2}{(\omega\tau)^2+1}e^{-\frac{t}{\tau}} \tag{9}$$

Applying a trigonometric identity, (9) becomes

$$y(t) = A\left(\frac{K}{\sqrt{(\omega\tau)^2+1}}\right)\sin(\omega t - \arctan(\omega\tau)) + A\left(\frac{K\omega\tau^2}{(\omega\tau)^2+1}\right)e^{-\frac{t}{\tau}} \tag{10}$$

Assuming the pole of $G(s)$ is in the left-half plane, τ will be positive, and the exponential term will vanish as t becomes large. The steady state response of the system can then be written as

$$y(t) = A|G(j\omega)|\sin(\omega t + \phi), \qquad \phi \equiv \angle G(j\omega) \tag{11}$$

This clearly shows that a sinusoidal input of frequency ω produces a sinusoidal output of frequency ω. The amplitude of the output is scaled by a factor of |G(jω)| and the phase lags behind the input by ∠G(jω). For the standard first-order system in (5), given the values of K and τ, the "gain" |G(jω)| and the "phase" ∠G(jω) can be expressed as a function of ω, as follows

$$|G(j\omega)| = \frac{K}{\sqrt{1+\omega^2\tau^2}} = \frac{K}{\sqrt{1+\frac{\omega^2}{\omega_B^2}}}; \qquad \angle G(j\omega) = -\arctan(\omega\tau) \tag{12}$$

where $\omega_B = 1/\tau$ is the bandwidth frequency.

For any transfer function $G(j\omega)$, we may plot the magnitude $|G(j\omega)|$ and phase angle $\angle G(j\omega)$ as functions of frequency ω in what is called a Frequency Response Diagram. The Amplitude ratio $|G(j\omega)|$ may be expressed in dimensional or dimensionless form. The latter is sometimes preferred. In practice the interesting range of $|G(j\omega)|$ may cover several orders of magnitude. For this reason it is often expressed on a logarithmic scale in decibels (dB). This can only be used when the $|G(j\omega)|$ has been made dimensionless, and the relevant definition is:

$$Gain(dB) = 20Log_{10}(|G(j\omega)|) \tag{13}$$

On a logarithmic scale $|G(j\omega)|$ and $\angle G(j\omega)$ can be plotted to generate what are known as gain and phase plots, or Bode diagrams. Typically gain is plotted as decibels (dB) and phase angle plotted in degrees. The Frequency Response diagrams for a standard first-order system are shown in Fig.3.

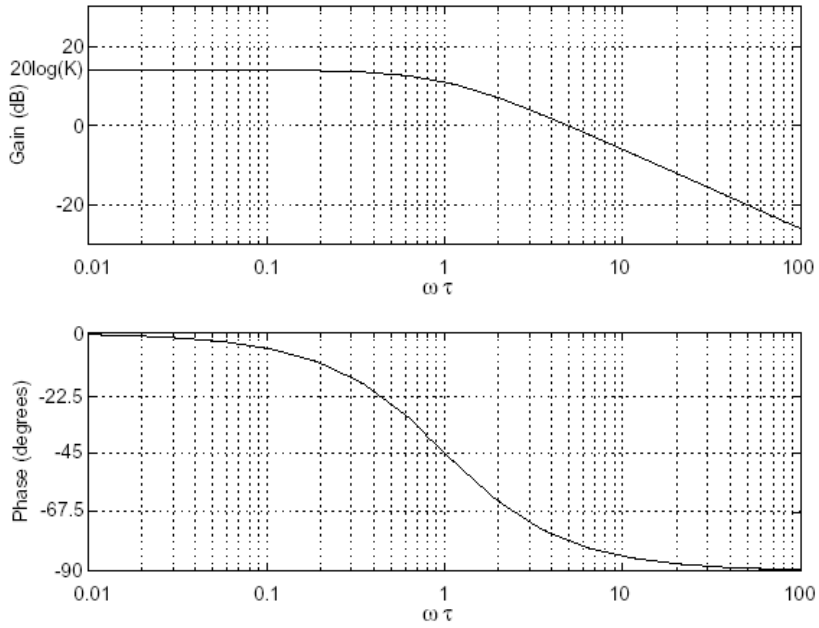


Figure 3. Frequency Response of a First-order system to a Sinusoidal Command.

Decibel Definition: The logarithmic "Bel" scale is named after Alexander Graham Bell and is the logarithm of the ratio of two power quantities. Power is always computed as the product of a constant and the square of a measured variable v.

$$P = kv^2 \tag{9}$$

The Bel is the logarithm of measured power and a reference power. In this case, the constant k always cancels.

$$\begin{aligned} B &= \log_{10}\left(\frac{P}{P_{ref}}\right) = \log_{10}\left(\frac{kv^2}{kv_{ref}^2}\right) \\ \Rightarrow\ B &= 2\log_{10}\left(\frac{v}{v_{ref}}\right) \end{aligned} \tag{10}$$

This scale is often too coarse for engineering use and traditionally we use *tenth* of Bel, the deciBel, denoted as dB in our computations. Because there are 10 dB per Bel,

$$dB = 10\left[2\log_{10}\left(\frac{v}{v_{ref}}\right)\right] = 20\log_{10}\left(\frac{v}{v_{ref}}\right) \tag{11}$$

When the reference for the measured signal is $v_{ref} = 1$, the dB computation simplifies to

$$dB = 20\log_{10}(v) \tag{12}$$

Pre-Lab Sample Questions

1) Convert the following values from linear scale to decibel scale:

a) 0.6 **b)** 10

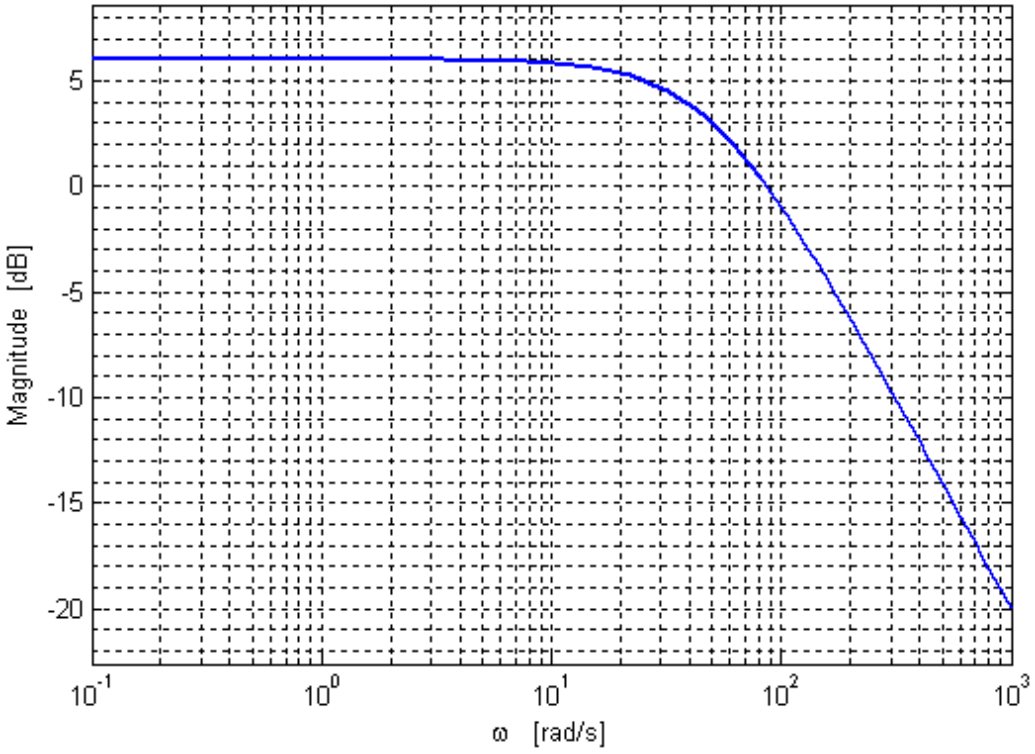
Convert the following values from decibel scale to linear scale:

c) 8 dB **d)** -23 dB

Answer: **a)** -4.4 dB **b)** 20 dB

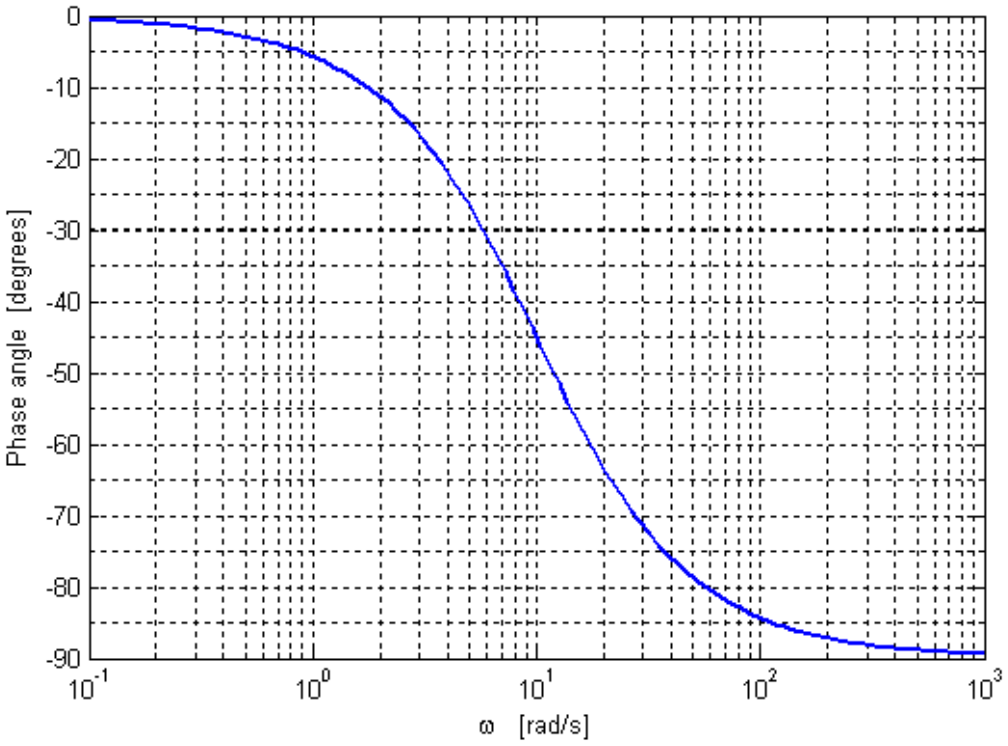
c) 2.5 **d)** 0.07

2) From the plot below, what is the non-dimensional DC Gain of the system?



Answer: DC Gain = 2

3) From the plot below, what is the time constant of the system?



Answer: $\tau = 0.1$ seconds

3. Experiment

PART A: Response of a first order plant

Questions to answer in the short form:

A.1 For given K = 1.56 and τ = 0.0033s, use Matlab to plot $20Log_{10}(|G(j\omega)|)$ and $\angle G(j\omega)$ versus ω for $1 \le \omega \le 10^5$. Attach the plot. The following Matlab code can be used for this purpose.

```
num=[1.56]; % Numerator of the transfer function = K = 1.56
den=[0.0033 1]; % Denominator of the transfer function= 1+τs
w=logspace(0,5); % Range of the frequency: 10^0 ≤ ω ≤ 10^5
H=freqs(num,den,w); % Complex Frequency Response, G(jω)
phase=angle(H)*180/pi; % Phase Angle (in degrees)= ∠G(jω)
mag=abs(H); % Gain( Magnitude)= |G(jω)|
dB=20*log10(mag); % Gain in decibels = 20Log10(|G(jω)|)
subplot(2,1,1); % First plot of the two
semilogx(w,dB); % x-axis (ω) in log scale
ylabel('Magnitude (dB)'); % labeling of y-axis
subplot(2,1,2); % Second  plot of the two
semilogx(w,phase); % x-axis (ω) in log scale
ylabel('Phase(degrees)'); % labeling of y-axis
xlabel('Frequency(rad/sec)'); % labeling of x-axis
```

Alternatively, the "bode" command may be used. Type "help bode" to learn more.

A.2. The plot obtained in A.1 illustrates the frequency response of a typical first order system. Studying this plot, explain what are the frequency response characteristics of a typical first order system at very low ($\omega \to 0$) and very high frequency ($\omega \to \infty$).

PART B: Analysis of a first order plant

B.I. A low-pass filter based on an operational amplifier circuit

Consider the operational amplifier circuit in Fig.4, with *V*1 and *V*6 as the input and output voltages, respectively. The system parameters include resistors *R*1, *R*2, and *R*3, and capacitor C.

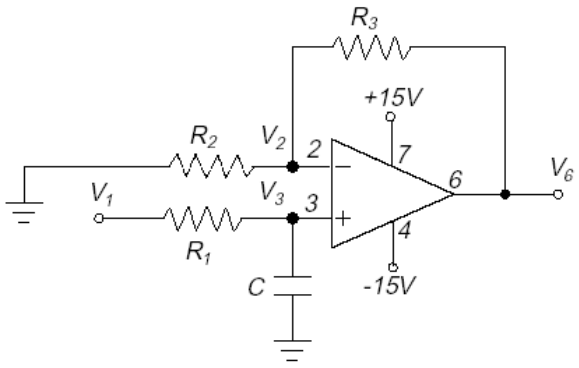


Figure 4. 741 Operational amplifier circuit.

An actual representation of the operational amplifier is shown in Figure 5.

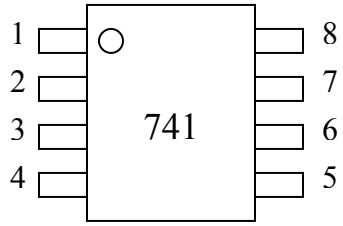


Figure 5. 741 Operational amplifier chip

When working with operational amplifiers, some basic assumptions can be made to aid in the analysis, these are the ideal op-amp assumptions:

$$\left.\begin{array}{ll} 1) & V_3 - V_2 \cong 0 \\ 2) & i_2 \cong 0 \\ 3) & i_3 \cong 0 \end{array}\right\} \qquad (13)$$

Assumption 1 says that the voltage at pin 2 is equal to the voltage at pin 3. Assumptions 2 and 3 say that there is no current entering or leaving the op-amp chip through pins 2 and 3.

Recall:

Kirchoff's Current Law: The sum of all currents flowing into a node equals zero.

Kirchoff's Voltage Law: The sum of all voltage drops/rises around a loop is zero.

Ohm's Law: V=Ri where V=voltage, R=resistance, i=current

Questions to answer in the short form:

B.1. Use the basic assumptions for the ideal operational amplifier as given above (13) and compute the transfer function for the circuit as shown in Figure 4. Does your finding/analysis tell you if this plant is suitable for analysis for this experiment?

Note: Using complex impedances in the frequency domain is easier than the time domain, so the capacitance C can be rewritten as $\frac{1}{sC}$ and modeled as a resistor.

**** Before you proceed, check your answer with your Lab Consultant.**

B.2. Find the values for R_1, R_2, R_3, and C to get K = 1.56 and τ = 0.0033s.

B.II. Experimental Setup

For this experiment you will be provided with the following equipment:

1. Power supply PS150E
   This unit provides DC voltages in the range of +15 V and -15 V.
2. Electronic components
   (a) Operational amplifier (OP-Amp 741), (b) Resistors, and (c) Capacitors
3. Universal breadboard
   The breadboard provides a reusable method of using electronic components. It provides a means for connecting different electronic components using jumper wires and eliminates the need for soldering.
4. Oscilloscope
   The oscilloscope will be used to measure and display voltage signals as functions of time
5. Signal Generator
   This unit will be used to generate a sinusoidal signal of a given frequency and amplitude.
6. Matlab script “tsfreq.m"
   The script generates Frequency Response plots using data obtained from experiments.

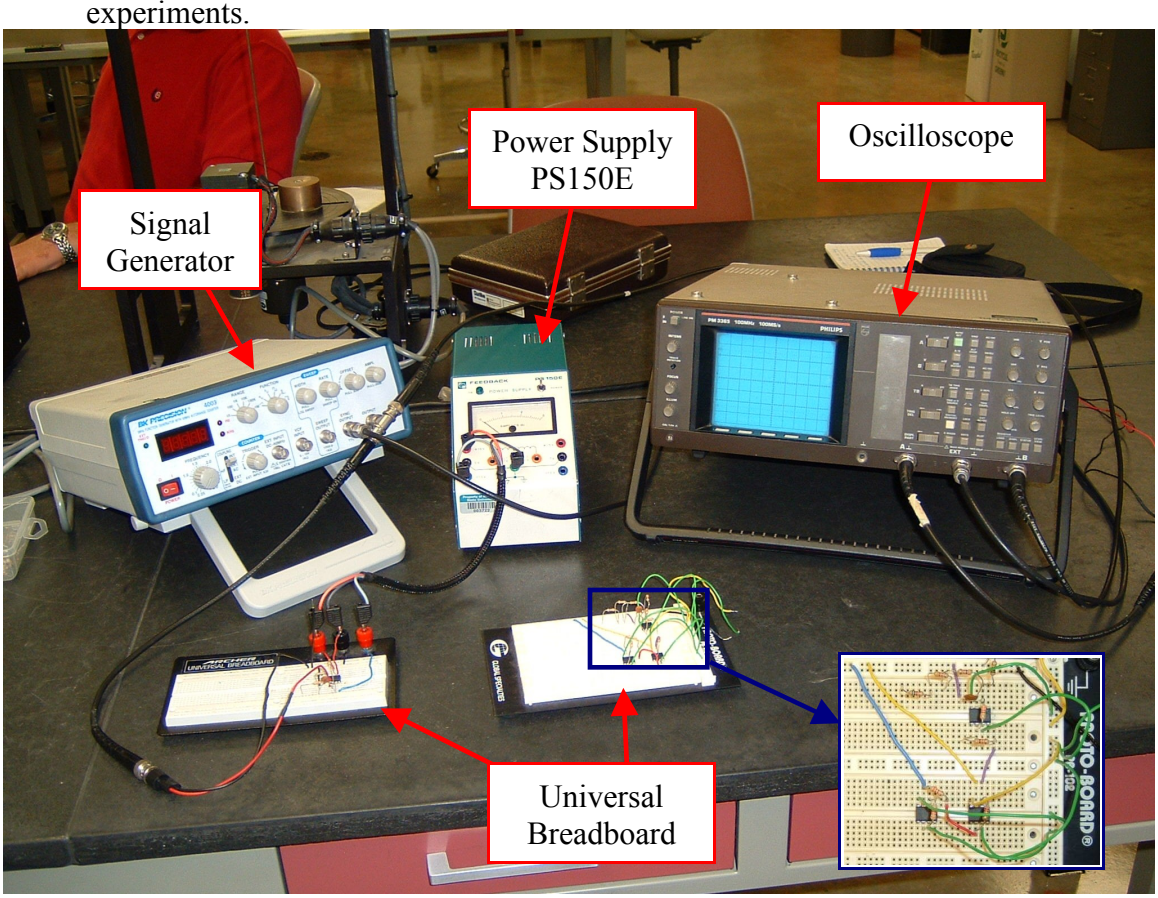


Figure 6 – Basic Equipment Setup

Circuit Construction:

1. **With the power supply, oscilloscope, and signal generator OFF**, build the circuit in Fig. 4 using the values of R1, R2, R3 and C you obtained in A.2.
2. Use the power supply to provide ±15 V and ground for the circuit.
3. Connect the signal generator to the circuit input at node V1. Connect signal generator and circuit grounds properly
4. Make connections to observe output (from node V6) on channel A of the Oscilloscope.
5. Make connections to observe input (that is coming from signal generator) on channel B of the Oscilloscope.

B.III. Steady state response to sinusoidal input, Experimental Procedures

In this experiment you will analyze the response of the electronic circuit in Fig.4 to a sinusoidal input.

1. **Turn ON the oscilloscope and the power supply.**

2. **Turn ON the signal generator** and set "Mode Function" to Sine, "Freq" to 10 Hz, "Amplitude" to 1 Vpp, and "Offset" to 0 V. You should notice both input (command) and output (response) sinusoidal signals on the oscilloscope.

3. Find output to input ratio, V6/V1 and the phase lag ϕ between the input and the response signals.

4. Repeat step 3 for frequencies provided in Table 1. Record all results in Table 1. For accurate reading from the oscilloscope, you may need to rescale the horizontal axis such that 2 to 3 cycles are seen on the screen.

5. From the course website, save the file "tsfreq.m" to the desktop. MAKE SURE YOU SAVE THE FILE EXTENSION (.m). Run the Matlab script "tsfreq.m" and enter the values from Table 1 to obtain the Frequency Response plots.

Questions to answer in the short form:

B.3. From the data obtained, determine the DC gain and time constant of the system. Compare these values with their theoretical values. Explain difference in the values, if any.

PART C: Modeling the first order DC Servomotor using an Operational Amplifier Circuit

Design Problem:

Design the op-amp circuit of Fig. 4 to model the DC servomotor from experiment 1. Suppose the DC gain (K) and time constant (τ) of the first order motor system was found to be 0.8 and 0.01s, respectively.

Questions to answer in the short form:

C.1. Using only those values of resistors and capacitors provided in the component box, what is the possible range of values for the DC gain and time constant?

C.2. Are your design values included in that range? (YES/NO) If not, figure out a way to get around this. (Hints: Multiples of the desired gain can be used and scaled with subsequent stages. Resistors put in series with R1 (fig.4) can increase the time constant)

C.3. Draw the designed op-amp circuit. (Show the designed values of the components).

C.4. What will be the theoretical values of the system's gain ($|G(j\omega)|$) and phase lag ($\angle G(j\omega)$) at frequencies, ω = 100rad/sec and 1000rad/sec?

PART D: Verification of the Design

Construct the op-amp circuit that you designed in part C. With the circuit that you constructed (from your design), you have to experimentally determine the value of DC gain (K) , time constant (τ), and bandwidth (ω_B).

Questions to answer in the short form:

D.1. Calculate the experimental values of the system's gain ($|G(j\omega)|$) and phase lag ($\angle G(j\omega)$) at frequencies, ω = 100 and 1000rad/sec (note: the signal generator input is f (cyc/sec)). Do they match with the corresponding values in C.4?

D.2. Calculate the experimental values of K, τ and ω_B from the known experimental values of $|G(j\omega)|$ and $\angle G(j\omega)$ in D.1. Do they match with their theoretical values?

Laboratory Report

Name: Section: Date:

A.1 For given K = 1.56 and τ = 0.0033s, use Matlab to plot $20Log_{10}(|G(j\omega)|)$ and $\angle G(j\omega)$ versus ω for $1 \le \omega \le 10^5$. Attach the plot.

A.2 The plot obtained in A.1 illustrates the frequency response of a typical first order system. Studying this plot, explain what are the frequency response characteristics of a typical first order system at very low ($\omega \to 0$) and very high frequency ($\omega \to \infty$).

B.1. Using the basic assumptions for an ideal operational amplifier as given in (13), compute the transfer function for the circuit as shown in Figure 4.
Does your finding/analysis tell you if this plant is suitable for analysis for this experiment?

Attach the additional page with your work.

** Before you proceed, check your answer with your LAB CONSULTANT.

B.2. Find the values for R_1, R_2, R_3, and C to get K = 1.56 and τ = 0.0033s.

B.3.A

Table 1. Experimental data for Frequency Response plots of electrical circuit

Run#	Frequency (Hz)	V_6/V_1	ϕ (deg *rees*)
1	1		
2	5		
3	10		
4	20		
5	50		
6	100		
7	200		
8	500		
9	1000		
10	2000		

B.3.B. From the data obtained, determine the DC gain and time constant of the system. Compare these values with their theoretical values. Explain difference in the values, if any. (Attach the frequency response plots).

C.1. Using only those values of resistors and capacitors provided in the component box, what is the possible range (max/min) of values for the DC gain and time constant?

C.2. Are your design values included in that range? (YES/NO) If not, figure out a way to get around this. (Hints: Multiples of the desired gain can be used and scaled with subsequent stages. Resistors put in series with R1 (fig.4) can increase the time constant)

C.3. Draw the designed op-amp circuit. (Show the designed values of the components).

C.4. What will be the theoretical value of the system's gain ($|G(j\omega)|$) and phase lag ($\angle G(j\omega)$) at frequencies, ω = 100rad/sec and 1000rad/sec?

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D.2. Calculate the experimental values of K, τ and ω_B from the known experimental values of $|G(j\omega)|$ and $\angle G(j\omega)$ in D.1. Do they match with their theoretical values?