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ME451 Laboratory

Time Response Modeling and Experimental Validation of a Second Order Plant: Mass-Spring Damper System



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1. Objective

Linear time-invariant dynamical systems are categorized under first-order systems, second-order systems, and higher-order systems. A large number of second-order systems are described by their transfer function in “standard form”. This form enables us to investigate the response of a large variety of second-order systems for any specific input. The qualitative response depends primarily on the natural frequency, ω_n , and the damping ratio, ζ . Both ω_n and ζ are functions of system parameters. The objective of this experiment is to model a standard second-order system and to investigate the effect of system parameters and feedback on its response to a step input.

For this lab, we choose to experiment with a torsional mass-spring-damper system. It will be shown that ideally this system acts as a second order, time-invariant system. The transfer function, specifically the gains, natural frequency, and damping ratio, of the system will be experimentally obtained. Feedback will be used to vary the effective damping ratio and natural frequency.

2. Background

2.1. Second-order systems

The standard form of transfer function of a second-order system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (1)$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the output and input functions, respectively, ω_n is the natural frequency, and ζ is the damping ratio. For a unit step input ($U(s) = 1/s$), the response of the system, in Laplace domain, can be written as

$$Y(s) = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Assuming poles of $G(s)$ are complex ($\zeta < 1$) and the DC gain $G(0) = K = 1$, the time domain unit step response can be written as

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\beta\omega_n t + \theta), \quad \beta \equiv \sqrt{1 - \zeta^2}, \quad \theta \equiv \arctan(\beta / \zeta) \quad (2)$$

The step response of a second-order system with varying damping ratios is shown in Fig. 1. The plot is normalized by frequency and amplitude. The response of the system is oscillatory, but damped. The amount of damping is characterized by the damping ratio ζ , and the frequency of damped oscillation is $\beta\omega_n$. Note that the frequency of damped oscillation is different from the natural frequency ω_n . The settling time of a second order system is defined as the time it takes for the system to settle within 2% of its steady state value. It is given by the expression $T_s = 4\tau$, where τ is the time constant, and is given as $\tau = 1/\zeta\omega_n$. Notice how the time constant appears in the exponential term of eq. (2). The peak time, T_p , is the time required for the output to reach its most extreme value, and is given as

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Another useful response characteristic, Percentage overshoot, is defined as the ratio of the difference $y_{peak} - y(\infty)$ to $y(\infty)$, where y_{peak} is the amplitude of the first peak. Unlike the damped

natural frequency, settling time, and peak time, it is independent of ω_n ; it is only a function of ζ . The following relation gives percentage overshoot:

$$\text{percentage overshoot} = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100$$

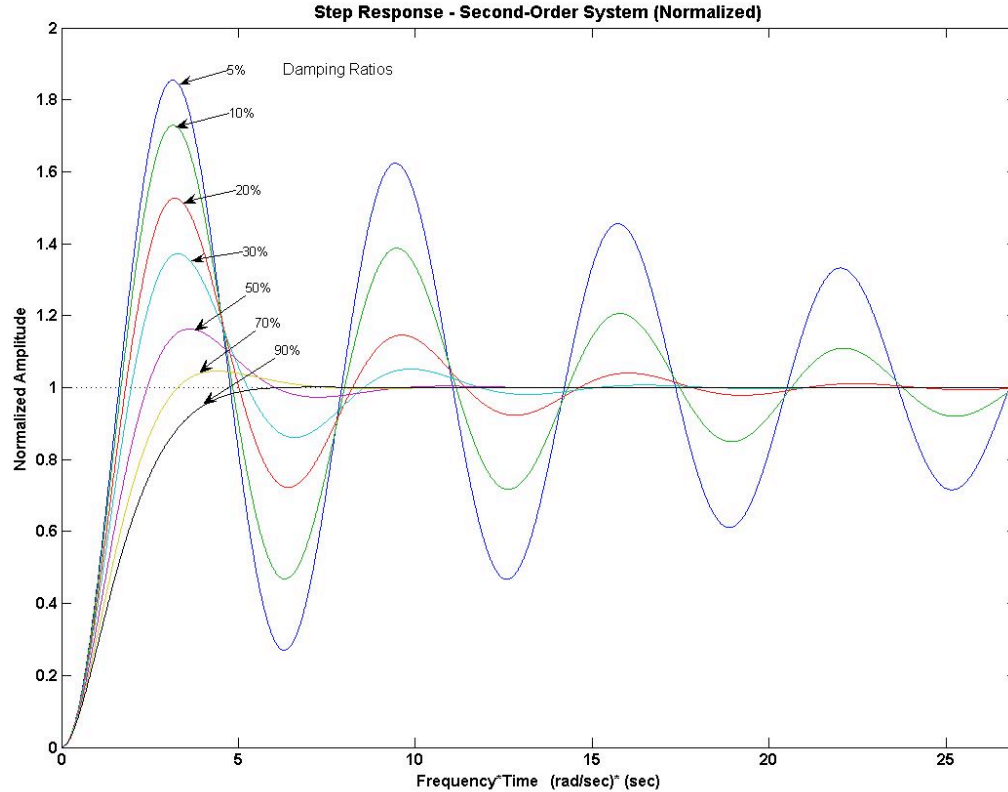


Figure1. Normalized Step Response of a Second Order System

In the event that the poles of $G(s)$ are real ($\zeta \geq 1$), the time domain response can be critically damped ($\zeta = 1$), or overdamped ($\zeta > 1$). For $\zeta = 1$, the time domain response of the output takes the form

$$y(t) = 1 + k_1 e^{-t/\tau} + k_2 t e^{-t/\tau} \quad (3)$$

where k_1 and k_2 are constants and τ is the time constant corresponding to the repeated pole of $G(s)$.

For $\zeta > 1$, the time domain response takes the form

$$y(t) = 1 + k_1 e^{-t/\tau_1} + k_2 e^{-t/\tau_2} \quad (4)$$

where τ_1 and τ_2 are the time constants corresponding to the real distinct poles of $G(s)$. Both Eqs. (3) and (4) depict cases where the response is not oscillatory, and hence, damped natural frequency, settling time, and peak time are not relevant. When $G(s)$ has real distinct poles ($\zeta > 1$) the time constant of the response is governed by the pole closer to the origin and the response resembles that of first-order systems.

2.2. Torsional mass-spring-damper system

Consider the torsional mass-spring-damper system in Fig.2. The system variables are

T external torque applied on rotor.

θ angular position of rotor.

ω angular velocity of rotor.

The parameters of the system, shown in Fig.2, include

J moment of inertia of rotor.

b coefficient of viscous friction.

k spring constant.

The transfer function of the mass-spring-damper system, defined with external torque as input and angular position of the rotor as output, can be written as

$$G(s) = \frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + bs + k} = \frac{1}{k} \left[\frac{(k/J)}{s^2 + (b/J)s + (k/J)} \right] \quad (5)$$

Figure 2(b) shows the experimental setup that is modeled with the diagram in Figure 2.

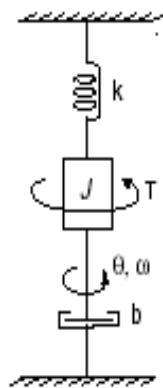


Figure 2: Diagram of torsional MSD system

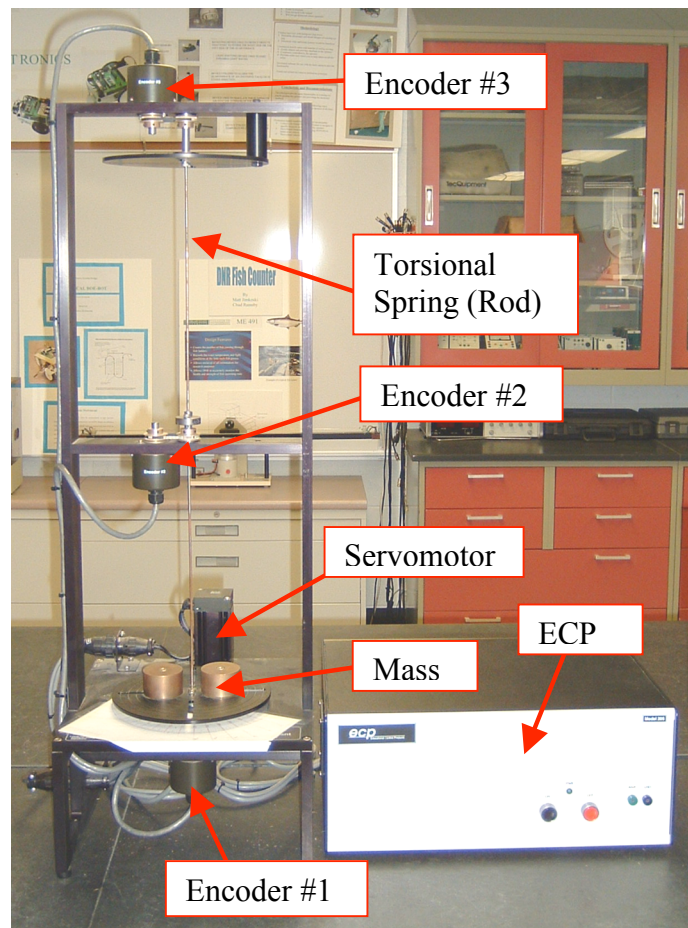


Figure 2(b) – Experimental Setup

The transfer function above bears close resemblance with the standard second-order transfer function in Eq.(1). The only difference is the DC gain of $(1/k)$, which appears in Eq.(5).

By comparing Eqs.(1) and (5), the expressions for natural frequency and damping ratio can be obtained as

$$\omega_n^2 = \frac{k}{J} \text{ and } 2\zeta\omega_n = \frac{b}{J} \text{ so that } \zeta = \frac{b}{2\sqrt{kJ}} \quad (6)$$

It is clear from Eq.(6) that ω_n and ζ are functions of system parameters J , b , and k , which are typically fixed. A physical implementation, with driving motor, of the torsional system is available for this laboratory and has the block diagram shown in Figure 3.

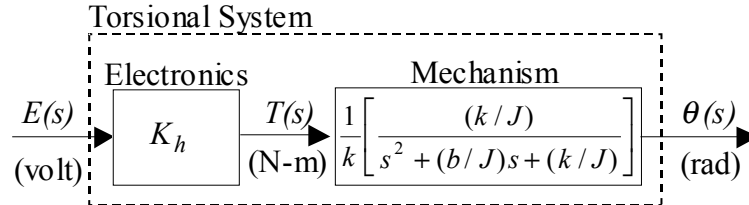


Figure 3: Block Diagram of the Open-Loop Torsional Mass-Spring-Damper System

This system has a transfer function closely related to the idealized transfer function (5) with the addition of a motor to provide input torque. This torsional system has a motor hardware gain, K_h (N-m/volt) that gives the torque to voltage ratio of the motor drive. If we build the system in Fig.3, we will have a specific second-order response of the form.

$$\frac{\theta(s)}{E(s)} = K_h \frac{\theta(s)}{T(s)} = K_h G(s) = \frac{K_h}{Js^2 + bs + k} = \frac{K_h}{k} \left[\frac{(k/J)}{s^2 + (b/J)s + (k/J)} \right] \quad (7)$$

The natural frequency and damping ratio for this system are the same as (5) with a revised DC gain of K_h/k .

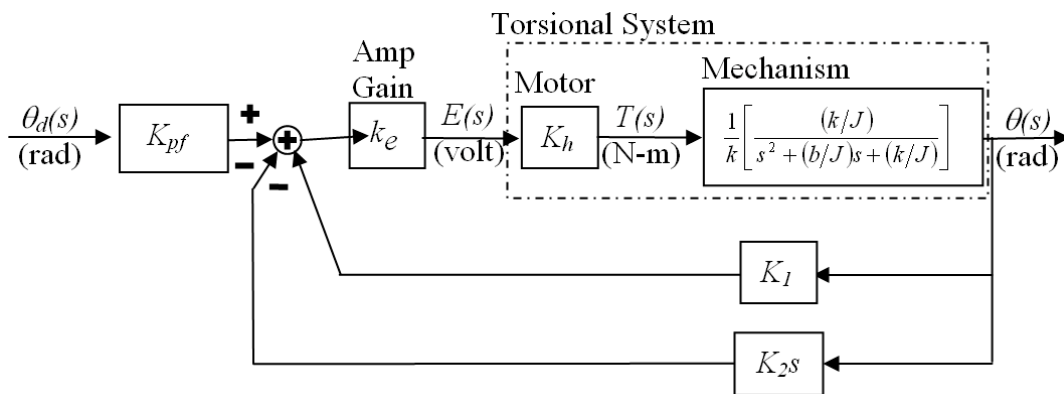


Figure 4. Block Diagram of the Closed-Loop Torsional Mass-Spring-Damper System

For studying the effect of system parameters on the response, we must be able to change the apparent J , b , and k . Consider the system shown in Figure 4. This closed-loop system achieves programmable changes in system damping and stiffness. We achieve this by

programming the motor to generate the torques generated by an additional spring and damper thereby changing the net stiffness and damping of the system. Specifically, the motor is programmed to generate the torque given by the relation

$$T = K_h k_e (K_{pf} \theta_d - K_1 \theta - K_2 \dot{\theta}) \quad (8)$$

where K_h is the electromechanical gain of the motor, k_e is the electronic gain of the DSP system, and θ_d is the desired output. Note that $K_1 \theta$ is the type of torque generated by a spring, and $K_2 \dot{\theta}$ is the type of torque generated by a damper. When these torques are applied on the rotor, the feedback generates different apparent stiffness and damping and the dynamic equation of the programmed system becomes

$$J\ddot{\theta} + b\dot{\theta} + k\theta = T \quad \Rightarrow \quad J\ddot{\theta} + b\dot{\theta} + k\theta + K_h k_e K_2 \dot{\theta} + K_h k_e K_1 \theta = K_h k_e K_{pf} \theta_d$$

Taking the Laplace Transform of this equation gives us

$$Js^2\theta + bs\theta + k\theta + K_h k_e K_2 s\theta + K_h k_e K_1 \theta = K_h k_e K_{pf} \theta_d$$

The block diagram of the programmed system is shown in Fig.4. The transfer function of this system, with T as input and θ as output, is

$$G(s) = \frac{\theta(s)}{\theta_d(s)} = \frac{[K_h k_e K_{pf}]/J}{[(k + K_h k_e K_1)/J]} \left[\frac{[(k + K_h k_e K_1)/J]}{s^2 + [(b + K_h k_e K_2)/J]s + [(k + K_h k_e K_1)/J]} \right] \quad (9)$$

which has the same structure as that in Eq.(1). The DC gain of the programmed system is

$$K = [K_h k_e K_{pf}]/(k + K_h k_e K_1),$$

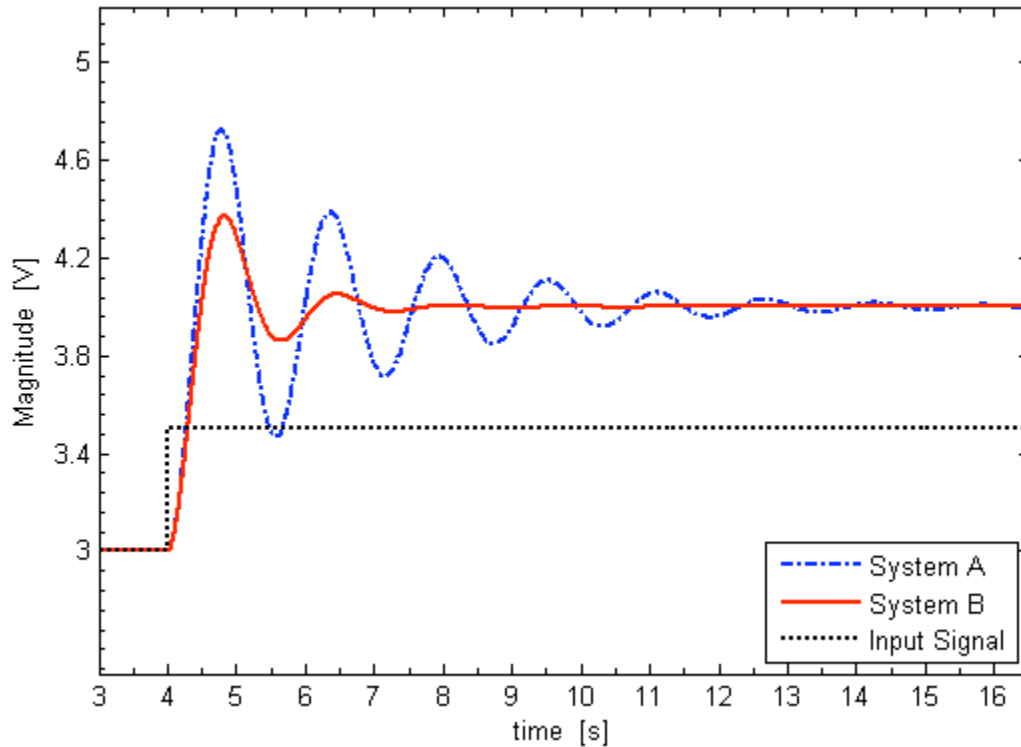
and the natural frequency and damping ratio are given by the relations

$$\omega_n^2 = (k + K_h k_e K_1)/J \text{ and } 2\zeta\omega_n = (b + K_h k_e K_2)/J \text{ so that } \zeta = \frac{(b + K_h k_e K_2)}{2\sqrt{(k + K_h k_e K_1)J}} \quad (10)$$

One of the objectives of this experiment is to study the effects of varying the natural frequency ω_n and the damping ratio ζ on response of the torsional mass. We will vary ω_n and ζ by varying the gains K_1 and K_2 in software. The hardware gain $K_h k_e$ and the rotor inertia J will remain fixed during experiments.

Pre-Lab Sample Questions

Use the plot below to answer the following questions:



- 1) Which system has a higher damping ratio?

Answer: System B

- 2) What is the percent overshoot of System A?

Answer: % OS $\approx 70\%$

- 3) What is the peak time of System B?

Answer: $T_p \approx 0.8$ s

- 4) What is the natural frequency of System A?

Answer: $\omega_n \approx 4.19$ rad/s

- 5) What is the steady-state gain of System B?

Answer: Steady-state gain = 2

Read Instructions Carefully!!!

3. Description of Experimental Setup

3.1. Hardware and software

1. Electromechanical plant, ECP Model 205

The electromechanical plant is comprised of the torsion disk, a DC servo motor that provides an external torque to the disk, and sensors for measurement of angular position of the disk.

2. ECP input/output electronics unit, Model 205

It contains the power supply unit for the DC motor and associated electronic hardware.

3. Digital signal processor (DSP)

The DSP, installed in the PC, takes the sensor signals in digital form, provided by the analog-to-digital convertors (ADC), and computes the torque to be generated by the motor. The digital torque signal is sent to the motor via the digital-to-analog convertors (DAC).

4. Software

ECP executive program in Windows NT environment.

3.2. Basic setup

1. Place two 500 gm masses on the lowest disk at equal distances from the center. Verify that the masses are secured to the disk.

2. Run the program "Ecp32", in the ECP folder. From the "File" pull down menu click on "Load Settings" and choose "me451lab2.cfg". This file should be downloaded from the software section of the lab website. Right-click on the file and save it to your desktop. Make sure that the control loop status is "Open" and the controller status is "OK". If not, click on the "Utility" pull down menu and select "Reset Controller". Next, click on the "Abort Control" icon with the red hand in the lower right corner of the screen. In the "Setup" pull down window, select "User Units...", then select "Radians" and "OK".

Caution: In this work, and all future work, make sure to stay clear of the mechanism unless otherwise directed. If the system appears to react violently, you should immediately click the "Abort Control" button on the ECP program window. If the problem persists, promptly turn off the ECP input/output electronic unit by pressing the red button on the electronics unit.

4. Experimental Procedures

Objective: In this experiment you will analyze the open-loop and closed-loop response of the torsional mass-spring-damper system to a step input. You will also vary the system gains and predict the change in response.

Part A: Test the open-loop system with no servo motor drive...

- a) **Turn the ECP electronics unit OFF,**
- b) Displace the disk (rotate) and observe it's behavior **Caution:** Do not displace the disk more than 20° because you can damage the long slender vertical shaft on this system by displacing it too far

→ *“Be nice to our shaft and it will be nice to you...”*

Questions to answer in the short form:

A.1. Estimate f_n (natural frequency in hertz) and compute ω_n (natural frequency in rad/sec). Estimate a range of ζ by comparing the system response to Figure 1.

Note: It is important to compare the natural frequency of the system to that of the plot before estimating ζ .

Part B: Test the open-loop system using servo motor torque

- a) **Turn the ECP electronics unit ON.** The block diagram shown in Fig. 3 now models the system.
- b) In the “Setup” pull down menu, choose “Control Algorithm”.
 - i) In the new window that opens, select “Continuous Time” for type, and “State Feedback” for control algorithm.
 - ii) Click on the box “Setup Algorithm” and set all the gains to zero. Click “OK”.
- c) Back in the “Setup Control Algorithm” window
 - i) click “Implement Algorithm”. The “Control Loop Status” is now “CLOSED” and all encoder readings are zero. Even though the “Control Loop Status” is “CLOSED”, setting all feedback gains to zero effectively makes it open-loop. Click “OK”.
- d) From the “Command” pull down menu, select “Trajectory”.
 - i) In the new window that opens, select “Step”.
 - ii) Click “Setup” to continue. Another window will open.
 - iii) Select the “Open Loop Step”. Use the default step size of 0.5 V, the dwell time to 2000 msec. and reps to 1. This will take you back to the previous window. You need to click “OK” again.
 - iv) Go to “Data”, “Setup Data Acquisition” and add “Control Effort” to the “selected items” list, click OK.
- e) From the “Command” pull down menu, click on “Execute”. Choose “Normal Data Sampling” then click on “Run”.
- f) You should immediately notice the disk turning. Wait for approximately 15 seconds for the disk to return to its original configuration and for data uploading. After the data has been successfully uploaded, a new window will open. Click “OK” in this window to accept the disc displacement measurement data.

- g) From the "Plotting" pull down menu, select "Setup Plot" and make sure "Control Effort" and "Encoder 1-Position" are listed in the left axis display, then click on "Plot Data" to display the system's response on the screen.
- h) From the "Plotting" pull down menu,
 - i) select "Axis Scaling". In the new window that opens, mark all "Plot Attributes" (this enables horizontal and vertical grids and plots all data points with visibly large circles making them more visible when printed) and click "OK".
 - ii) Select "Print Plot" under "Plotting" pull down menu to get a hard copy.

Questions to answer in the short form:

B.1. From your values obtained from your open-loop plot, compute ω_n and damping ratio ζ . Do they agree with question (1)? Why or why not?

B.2. From the open-loop plot compute the open-loop system's steady state Gain.

Part C: Measure Open-Loop System Parameters

Turn the ECP electronics unit OFF,

- a) Measure the torsional stiffness k of the torsion system
Using the electronic force indicator, apply a force at right angles to a radius of the disc and record the angular deflection of the disc. Record the force and radial distance at which it was applied. Compute the angular stiffness k of the torsion system in units of (N-m)/rad. Record your result for k .
- b) Find the hardware gain of the servo motor K_h
Using the DC gain computed in part B, your computed stiffness, and the Open-loop model shown in equation (7), find the hardware gain of the servo motor K_h
- c) Compute b, J of the system using Open Loop natural frequency and damping ratio along with your computed angular stiffness k .
- d) Find the electronic gain k_e of the DSP system
With the ECP electronics box OFF, the DSP system functions but the servo motor drive does not. We will generate a signal θ_d causing a voltage E and measure the ratio to find gain k_e
 - i) In the "Setup" pull down menu, choose "Control Algorithm". In the new window that opens, select "Continuous Time" for type, and "State Feedback" for control algorithm. Click on the box "Setup Algorithm" and set the gains below. Click implement the algorithm.

$$K_{pf} = 0.1; \text{ with } K_1 = K_2 = K_3 = K_4 = K_5 = K_6 = 0$$
 - ii) Set the closed-loop input θ_d to a 4000 millisecond step of step size 0.1 radians and execute this trajectory. Look at the closed loop error on the computer screen (Control Effort) and note it's value. With the feedback terms all set to zero, the closed-loop error to input ratio equals the gain product $K_{pf}k_e$ (see figure 4). Compute $K_{pf}k_e$
 - iii) From $K_{pf}k_e$, compute the DSP electronic gain k_e

Questions to answer in the short form:

- C.1. Apply a known torque to the open-loop system, measure the angular deflection and compute the system stiffness k in units of (N-m)/rad. Show your work.
- C.2. Using the DC gain computed in part 2, your computed stiffness, and the Open-loop model shown in equation (7), find the hardware gain of the servo motor K_h . Show your work.
- C.3. Compute b and J of the system using open loop natural frequency, damping ratio and k .
- C.4 Find the electronic gain k_e of the DSP system. Show your work.

Part D: Test the Closed Loop Response of the System

To complete this section, you will measure the parameters for many of the components shown in Figure 4, measure the closed-loop response and finally the steady state gain.

- a) Turn the ECP electronics unit ON. Now the block diagram shown in Figure 4 models the system.
- b) In the "Setup" pull down menu, choose "Control Algorithm".
 - i) In the new window that opens, select "Continuous Time" for type, and "State Feedback" for control algorithm.
 - ii) Click on the box "Setup Algorithm" and set all the gains to the values below. Click "OK".

$$K_{pf} = K_1 = 0.1; K_2 = 0.002; K_3 = K_4 = K_5 = K_6 = 0$$

- c) Back in the "Setup Control Algorithm" window
 - i) click "Implement Algorithm".

The "Control Loop Status" is now "CLOSED" and all encoder readings are zero. Click "OK".
 - d) From the "Command" pull down menu, select "Trajectory".
 - i) In the new window that opens, select "Step".
 - ii) Click "Setup" to continue. Another window will open.
 - iii) Select the "Closed Loop Step".

Set the step size to 0.1 radians, the dwell time to 1000 msec. and reps to 1, click "OK".

This will take you back to the previous window. You need to click "OK" again.
 - e) From the "Command" pull down menu, click on "Execute".
- Choose "Normal Data Sampling" then click on "Run".
- f) You should immediately notice the disk turning. Wait for approximately 15 seconds for the disk to return to its original configuration and for data uploading. After the data has been successfully uploaded, a new window will open. Click "OK" in this window to accept the disc displacement measurement data.
 - g) From the "Plotting" pull down menu, select "Setup Plot". Plot commanded position and Encoder 1 position on the left vertical axis. Click on "Plot Data" to display the system's response on the screen.
 - h) From the "Plotting" pull down menu,

- i) select "Axis Scaling". In the new window that opens, mark all "Plot Attributes" (this enables horizontal and vertical grids and plots all data points with visibly large circles making them more visible when printed) and click "OK".
- ii) Select "Print Plot" under "Plotting" pull down menu to get a hard copy.

Questions to answer in the short form:

D.1. Determine the percentage overshoot (PO), settling time (T_s), peak time (T_p), natural frequency (ω_n) and damping ratio (ζ), from the graph you obtained from the closed-loop experiment. Label your graph and show all calculations.

D.2. Compute the steady state gain of the closed loop system using equation (9). How does this compare to the value determined from the plot?

Part E: Predicting System Response

Knowing k , b , J , K_h , k_e , you will predict and verify closed loop system response.

- a) Using your known values above, predict (calculate) the natural frequency and damping ratio of the system.
- b) Repeat steps D a-h using the gains given below.

$$K_{pf} = K_1 = 0.075; K_2 = 0.004; K_3 = K_4 = K_5 = K_6 = 0$$

Estimate the percentage overshoot (PO), peak time (T_p), natural frequency (ω_n) and damping ratio (ζ), from the graph you obtained. Label your graph and show all calculations. How well do these values compare to your predictions?

Questions to answer in the short form:

E.1. List and compare your calculated and estimated values for the natural frequency and damping ratio of the system.

Short Form Laboratory Report

Name:

Section:

Date:

A.1. Estimate values for f_n and ζ , and compute ω_n .

B.1. From your values obtained from your open-loop plot, compute ω_n and damping ratio ζ . Do they agree with question (1)? Why or why not?

B.2. From the open-loop plot compute the open-loop system's steady state Gain (1 rad = 1 volt)

C.1. Apply a known torque to the open-loop system, measure the angular deflection and compute the system stiffness k in units of (N-m)/rad. Show your work.

C.2. Using the DC gain computed in part 2. and your computed stiffness, and the Open-loop model shown in equation (7), find the hardware gain of the servo motor K_h . Show your work

C.3. Compute b and J of the system using open loop natural frequency, damping ratio and k .

Short Form Laboratory Report

Name:

Section:

Date:

C.4 Find the electronic gain k_e of the DSP system. Show your work.

D.1. Determine the percentage overshoot (PO), settling time (T_s), peak time (T_p), natural frequency (ω_n) and damping ratio (ζ), from the graph you obtained from the closed-loop experiment. Label your graph and show all calculations.

D.2. Compute the steady state gain of the closed loop system using equation (9). How does this compare to the value determined from the plot?

Short Form Laboratory Report

Name:

Section:

Date:

E.1. List and compare your calculated and estimated values for the natural frequency and damping ratio of the system.

Conclusion

Summarize the lessons you have learned from this laboratory experience, in a few sentences.