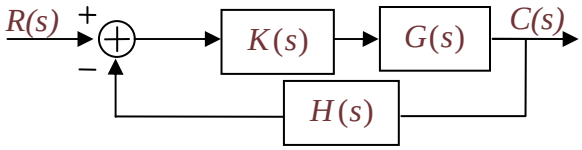
Root Locus Example

Consider the velocity control of a servo system



The control is a "PI" control with transfer function

$$K(s) = \frac{K_P s + K_I}{s} = \frac{K_P (s+2)}{s} \text{ where } \frac{K_I}{K_P} = 2$$

The plant is a first order system with transfer function

$$G(s) = \frac{1}{s+1} \text{ where the servo's time constant } \tau_{servo} = 1 \text{ sec}$$

The servo speed is measured with a speed sensor with transfer function

$$H(s) = \frac{4}{s+4} \text{ where the sensor's time constant } \tau_{sensor} = 0.25 \text{ sec}$$

The system's open-loop transfer function is

$$K(s)G(s)H(s) = K_P \frac{4(s+2)}{s(s+1)(s+4)} = k \frac{(s+2)}{s(s+1)(s+4)} \text{ where the parameter } k = 4K_P$$

Now draw the Root Locus using the rules…

1. There are 3 branches.

3. The real axis between –1<s<0 and –4<s<-2 is on the root locus because these areas are to the left of an odd number of poles and zeros.

4. The Root Loci start ($K = 0$) at the poles located at –4, -1 and 0 then end at the zero located at –2 as well as (2) zeros at $s = \infty$.

5. For large values of s, the Root Loci are asymptotic to **asymptotes** with angles,

$$\theta = \theta_a = \frac{(2k+1)180^O}{\|3-1\|} = 90^O, 270^O$$

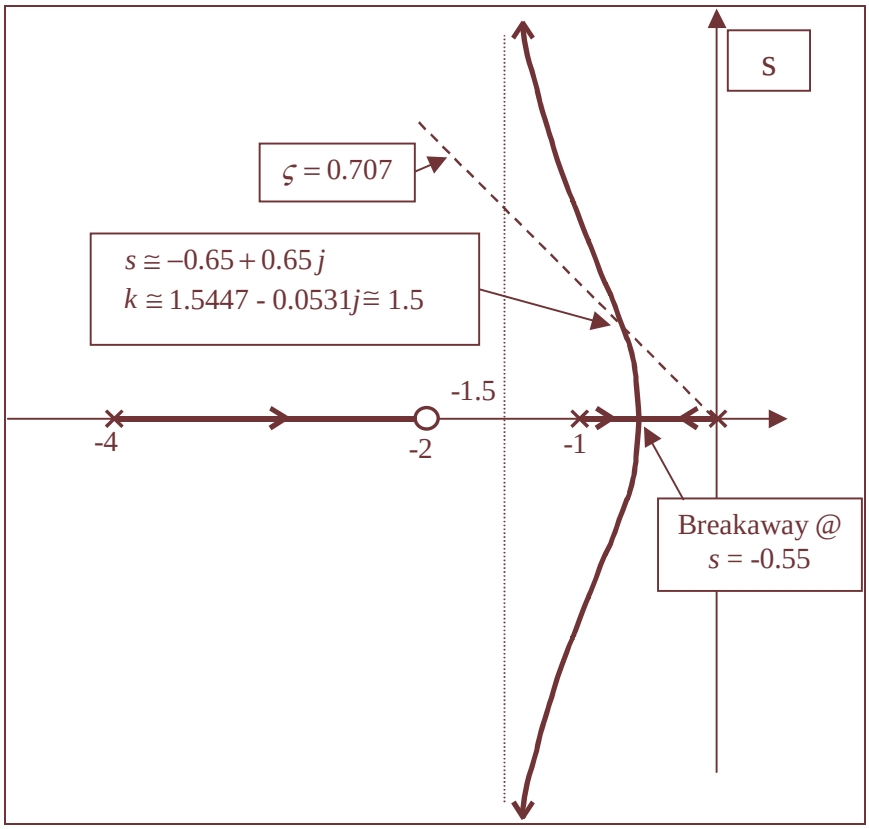
The **intersection of the asymptotes** lies on the real axis at

$$s = \sigma_a = \frac{\sum(0-1-4) - \sum(-2)}{3\text{-}1} = \frac{-5+2}{2} = \frac{-3}{2} = -1.5$$

6. A **Breakaway point** occurs along the root locus $-1 < s < 0$ at a relative maximum value of *k*.

Calculate $k = \dfrac{-D(s)}{N(s)} = \dfrac{-s(s+1)(s+4)}{(s+2)}$ and tabulate versus "*s*".

s	$k = \dfrac{-D(s)}{N(s)} = \dfrac{-s(s+1)(s+4)}{(s+2)}$	
-0.25	$k = \dfrac{-(-0.25)(-0.25+1)(-0.25+4)}{(-0.25+2)} = \dfrac{(0.25)(0.75)(3.75)}{(1.75)} = 0.402$	
-0.50	$k = \dfrac{-D(s)}{N(s)} = \dfrac{-s(s+1)(s+4)}{(s+2)} = \dfrac{(0.50)(0.50)(3.50)}{(1.50)} = 0.583$	
-0.75	$k = \dfrac{-D(s)}{N(s)} = \dfrac{-s(s+1)(s+4)}{(s+2)} = \dfrac{(0.75)(0.25)(3.25)}{(1.25)} = 0.488$	
-0.625	$k = \dfrac{-D(s)}{N(s)} = \dfrac{-s(s+1)(s+4)}{(s+2)} = \dfrac{(0.625)(0.375)(3.375)}{(1.375)} = 0.575$	
-0.6	$k = \dfrac{-D(s)}{N(s)} = \dfrac{-s(s+1)(s+4)}{(s+2)} = \dfrac{(0.6)(0.4)(3.4)}{(1.4)} = 0.583$	
-0.55	$k = \dfrac{-D(s)}{N(s)} = \dfrac{-s(s+1)(s+4)}{(s+2)} = \dfrac{(0.55)(0.45)(3.45)}{(1.45)} = 0.589$	max



Root Locus can be calibrated to find gain at important points. Here

$$K_P = \frac{k}{4} \approx \frac{1.5}{4} = 0.38$$

Matlab Solution:

$$K(s)G(s)H(s) = K_P \frac{4(s+2)}{s(s+1)(s+4)} = k \frac{(s+2)}{s(s+1)(s+4)} \text{ where the parameter } k = 4K_P$$

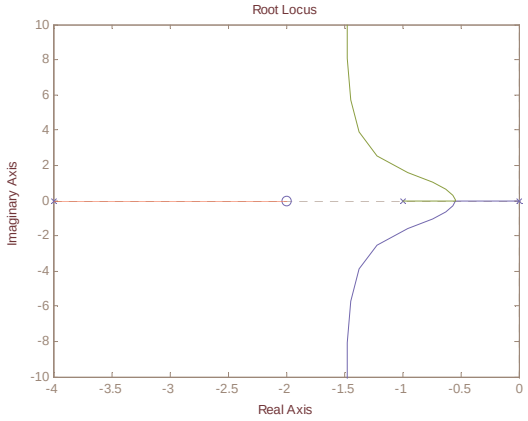
The Matlab commands to generate the root locus are: ("conv" multiplies (2) polynomials)

```
>> Num=[1 2];Den=conv([1 0],conv([1 1],[1 4]));
>> GH=tf(Num,Den)

Transfer function:
      s + 2
-----------------
s^3 + 5 s^2 + 4 s

>> rlocus(GH)
```

These Matlab Commands generate the graphic below that has "distorted" x-y axis scaling.



Using the Figure Properties editor to change the x-y axis scaling produces the "undistorted" root locus below. "Clicking" on the root locus drawn by Matlab causes it to echo the gain, damping ratio, etc. confirming the results of our sketch.

