ME451: Control Systems

Lecture 7b
Modeling of Rotating Mechanical Systems

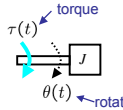
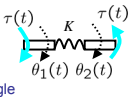
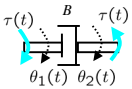
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Department of Mechanical Engineering
Michigan State University

Course roadmap

Modeling	Analysis	Design
✔ Laplace transform ✔ Transfer function ✔ Models for systems ✔ electrical • mechanical • electromechanical ✔ Block diagrams Linearization	Time response • Transient • Steady state Frequency response • Bode plot Stability • Routh-Hurwitz • Nyquist	Design specs Root locus Frequency domain PID & Lead-lag Design examples

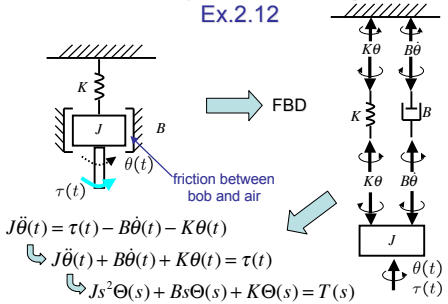
(Matlab simulations &) laboratories

Rotational mechanical elements (constitutive equations)

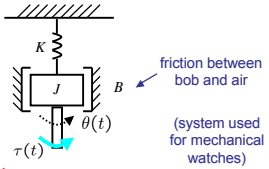
Moment of inertia	Rotational spring	Friction
		
$\tau(t) = J\theta''(t)$	$\tau(t) = K(\theta_1(t) - \theta_2(t))$	$\tau(t) = B(\theta_1'(t) - \theta_2'(t))$
$\begin{pmatrix}\theta(0)=0\\ \dot\theta(0)=0\end{pmatrix}$		$\begin{pmatrix}\Theta_1(0)=0\\ \Theta_2(0)=0\end{pmatrix}$
$T(s) = Js^2\Theta(s)$	$T(s) = K(\Theta_1(s) - \Theta_2(s))$	$T(s) = Bs(\Theta_1(s) - \Theta_2(s))$

Torsional pendulum system

Ex.2.12



Torsional pendulum system



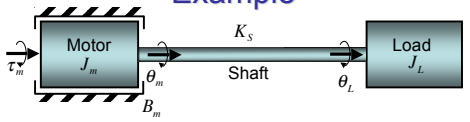
- Equation of Motion

$$J\ddot\theta(t) + B\dot\theta(t) + K\theta(t) = \tau(t)$$

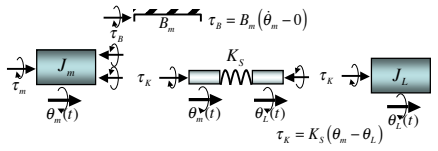
- By Laplace transform (with zero ICs),

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$ *(2nd order system)*

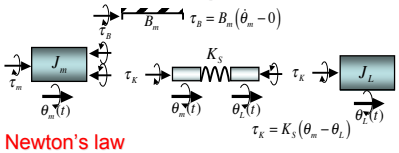
Example



- FBD



Example

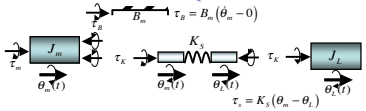


- By Newton's law

$$J_M\ddot{\theta}_m(t) = \tau_m - \tau_B - \tau_K$$
$$= \tau_m - B_m\dot{\theta}(t) - K(\theta_m - \theta_L)$$
$$J_L\ddot{\theta}_L(t) = \tau_K$$
$$= K(\theta_m - \theta_L)$$

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Example



- By Newton's law

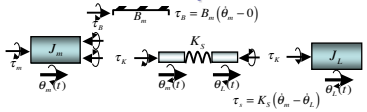
$$J_M\ddot{\theta}_m(t) = \tau_m - B_m\dot{\theta}(t) - K(\theta_m - \theta_L)$$
$$J_L\ddot{\theta}_L(t) = K(\theta_m - \theta_L)$$

- By Laplace transform (with zero ICs),

$$J_M s^2\Theta_m(s) + B_m s\Theta_m(s) + K(\Theta_m - \Theta_L) = T_m$$
$$J_L s^2\Theta_L(t) - K(\Theta_m(s) - \Theta_L(s)) = 0$$

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Example



- By Laplace transform (with zero ICs),

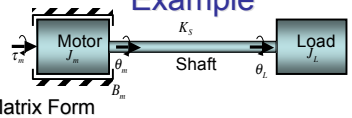
$$J_M s^2\Theta_m(s) + B_m s\Theta_m(s) + K(\Theta_m - \Theta_L) = T_m$$
$$J_L s^2\Theta_L(t) - K(\Theta_m(s) - \Theta_L(s)) = 0$$

- Matrix Form

$$\begin{bmatrix} (J_M s^2 + B_m s + K) & -K \\ -K & J_L s^2 + K \end{bmatrix} \begin{bmatrix} \Theta_m(s) \\ \Theta_L(s) \end{bmatrix} = \begin{bmatrix} T_m \\ 0 \end{bmatrix}$$

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Example



- Matrix Form

$$\begin{bmatrix} (J_M s^2 + B_m s + K) & -K \\ -K & J_L s^2 + K \end{bmatrix} \begin{bmatrix} \Theta_m(s) \\ \Theta_L(s) \end{bmatrix} = \begin{bmatrix} T_m \\ 0 \end{bmatrix}$$

- Solving for Transfer Function

$$\frac{\Theta_L(s)}{T_m} = \frac{\begin{vmatrix} (J_M s^2 + B_m s + K) & T_m \\ -K & 0 \end{vmatrix}}{\begin{vmatrix} (J_M s^2 + B_m s + K) & -K \\ -K & (J_L s^2 + K) \end{vmatrix}}$$
$$= \frac{K}{(J_M s^2 + B_m s + K)(J_L s^2 + K) - K^2}$$

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Example (cont'd)

- From second equation:

$$\Theta_L(s) = \underbrace{\frac{K}{J_L s^2 + K}}_{G_1(s)}\Theta_m(s)$$ *(2nd order system)*

- From first equation:

$$\Theta_m(s) = \underbrace{\frac{J_L s^2 + K}{s(J_m J_L s^3 + B_m J_L s^2 + K(J_m + J_L)s + B_m K)}}_{G_2(s)} T_m(s)$$

(4th order system)

Block diagram $T_m(s)$ → G2 → $\Theta_m(s)$ → G1 → $\Theta_L(s)$

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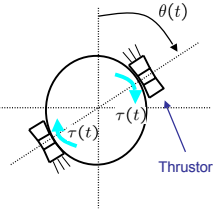
Satellite Picture



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Rigid satellite Ex. 2.13





- Broadcasting
- Weather forecast
- Communication
- GPS, etc.

$$\tau(t) = J\ddot{\theta}(t)$$

$$\Rightarrow \quad G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}$$

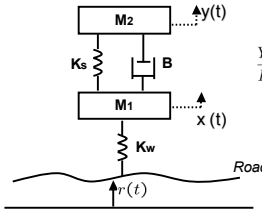
Double integrator

Summary & Exercises

- Modeling of mechanical systems
  - Translational
  - Rotational
- Next, Electromechanical systems

Exercises (Franklin et al.)

- *Quarter car model*: Obtain a transfer function from R(s) to Y(s).



Answer

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}$$