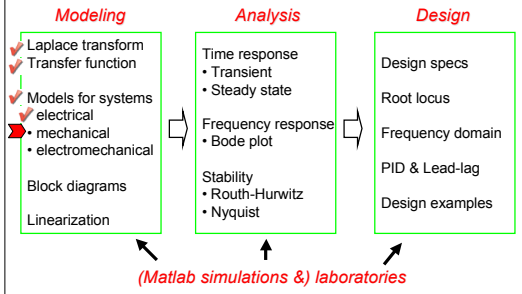
ME451: Control Systems

Lecture 7a
Modeling of Mechanical Systems

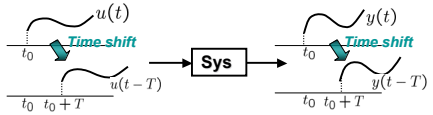
Prof. Clark Radcliffe and Prof. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Course roadmap



Time-invariant & time-varying

- A system is called ***time-invariant (time-varying)*** if system parameters do not (do) change in time.
- Example: Mx''(t)=f(t) & M(t)x''(t)=f(t)
- For time-invariant systems:

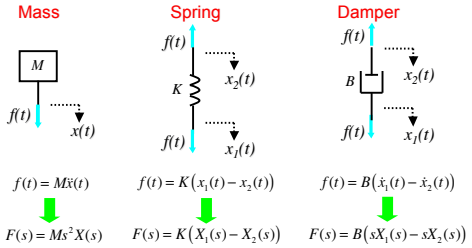


- This course deals with time-invariant systems.

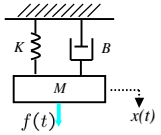
Newton's laws of motion

- 1st law:
  - A particle remains at rest or continues to move in a straight line with a constant velocity if there is no (unbalancing) force acting on it.
- 2nd law:
  - $\sum F_i(t) = m\frac{d^2x}{dt^2}$: translation
  - $\sum \tau_i(t) = I\frac{d^2\theta}{dt^2}$: rotation
- 3rd law:
  - For every action has an equal and opposite reaction

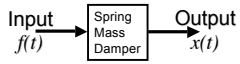
Translational mechanical elements: (constitutive equations)



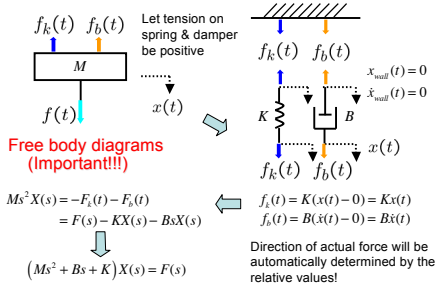
Mass-spring-damper system



- This system's transfer function is found by using Newton's Laws to relate the input force $f(t)$ to output displacement $x(t)$

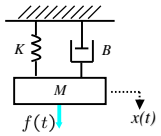


Newton's law: $M(d^2x/dt^2) = \Sigma(f)$



$$Ms^2X(s) = -F_k(t) - F_b(t)$$
$$= F(s) - KX(s) - BsX(s)$$
$$\Rightarrow \quad (Ms^2 + Bs + K)X(s) = F(s)$$

Mass-spring-damper system



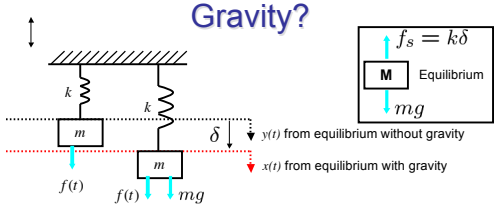
- Equation of motion (Laplace transformed, zero I.C.)

$$(Ms^2 + Bs + K)X(s) = F(s)$$

- The transfer function model (2nd order system)

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + Bs + K)}$$

Gravity?



- With gravity about y:

$$m\ddot{y} = f(t) + mg - ky \quad \Rightarrow \quad m\ddot{y} + ky = f(t) + mg$$

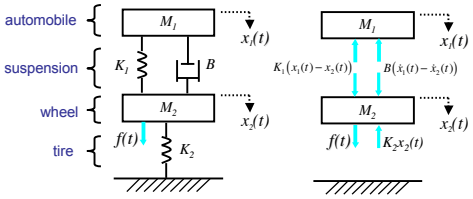
- y conversion to x coordinate:

$$y(t) = x(t) + \delta \qquad \ddot{y}(t) = \ddot{x}(t)$$

- With gravity about x:

$$m\ddot{x} + k(x + \delta) = f(t) + mg \quad \xrightarrow{k\delta = mg} \quad m\ddot{x} + kx = f(t)$$

Automobile suspension system



$$M_1\ddot{x}_1(t) = -B(\dot{x}_1(t) - \dot{x}_2(t)) - K_1(x_1(t) - x_2(t))$$
$$M_2\ddot{x}_2(t) = f(t) + B(\dot{x}_1(t) - \dot{x}_2(t)) + K_1(x_1(t) - x_2(t)) - K_2x_2(t)$$

Automobile suspension system

$$M_1\ddot{x}_1(t) = -B(\dot{x}_1(t) - \dot{x}_2(t)) - K_1(x_1(t) - x_2(t))$$
$$M_2\ddot{x}_2(t) = f(t) + B(\dot{x}_1(t) - \dot{x}_2(t)) + K_1(x_1(t) - x_2(t)) - K_2x_2(t)$$

Laplace transform with zero ICs

$$M_1s^2X_1(s) = -Bs(X_1(s) - X_2(s)) - K_1(X_1(s) - X_2(s))$$
$$M_2s^2X_2(s) = F(s) + Bs(X_1(s) - X_2(s)) + K_1(X_1(s) - X_2(s)) - K_2X_2(s)$$

Block diagram

$$\begin{cases} X_1(s) = \underbrace{\frac{Bs + K_1}{M_1s^2 + Bs + K_1}}_{G_1(s)} X_2(s) \\ X_2(s) = \underbrace{\frac{1}{M_2s^2 + Bs + K_1 + K_2}}_{G_2(s)} F(s) + \underbrace{\frac{Bs + K_1}{M_2s^2 + Bs + K_1 + K_2}}_{G_3(s)} X_1(s) \end{cases}$$



Automobile suspension system

- Laplace transformed Model

$$M_1s^2X_1(s) = -Bs(X_1(s) - X_2(s)) - K_1(X_1(s) - X_2(s))$$
$$M_2s^2X_2(s) = F(s) + Bs(X_1(s) - X_2(s)) + K_1(X_1(s) - X_2(s)) - K_2X_2(s)$$

- Matrix Model

$$\begin{bmatrix} (M_1s^2 + Bs + K_1) & (-Bs - K_1) \\ (-Bs - K_1) & (M_2s^2 + Bs + K_1 + K_2) \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}$$

- Linear Algebra Solution

$$\frac{X_2(s)}{F(s)} = \begin{vmatrix} (M_1s^2 + Bs + K_1) & 0 \\ (-Bs - K_1) & 1 \end{vmatrix} \Big/ \begin{vmatrix} (M_1s^2 + Bs + K_1) & (-Bs - K_1) \\ (-Bs - K_1) & (M_2s^2 + Bs + K_1 + K_2) \end{vmatrix}$$
$$= (M_1s^2 + Bs + K_1) \Big/ \left((M_1s^2 + Bs + K_1)(M_2s^2 + Bs + K_1 + K_2) - (Bs + K_1)^2 \right)$$

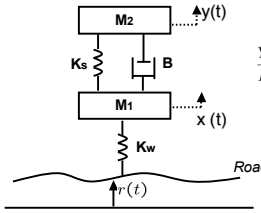
Summary & Exercises

- Modeling of mechanical systems
  - Translational
  - Rotational
- Next, rotational mechanical.
- Exercises
  - Read Sections 2.5, 2.6.
  - Derive equations for the automobile suspension problem.

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Exercises (Franklin et al.)

- *Quarter car model*: Obtain a transfer function from R(s) to Y(s).



Answer

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_W b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_W}{m_1}\right)s^2 + \left(\frac{k_W b}{m_1 m_2}\right)s + \frac{k_W k_s}{m_1 m_2}}$$

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