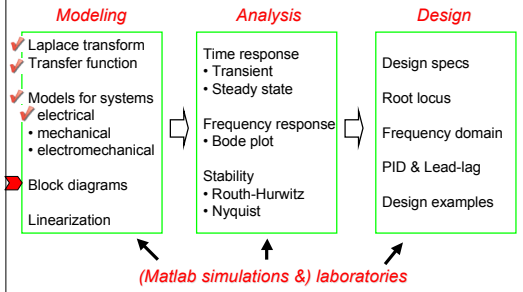
ME451: Control Systems

Lecture 6
Block diagrams

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Course roadmap



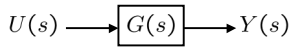
Transfer function (review)

- A transfer function is defined by

$$G(s) := \frac{Y(s)}{U(s)}$$

Laplace transform of system output

Laplace transform of system input

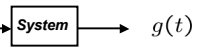


- A system is assumed to be at rest. (Zero initial condition)

Impulse response (review)

- Suppose that u(t) is the unit impulse function and system is at rest.

$$u(t) = \delta(t) \quad U(s) = 1$$

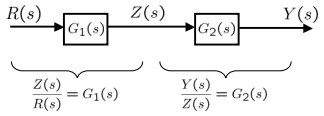


$$g(t)$$

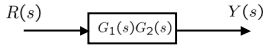
- The output g(t) for the unit impulse input is called ***impulse response***.
- Since U(s)=1, the transfer function can also be defined as the Laplace transform of impulse response: $G(s) := \mathcal{L}\{g(t)\}$

Elementary TF block diagrams

- Series connection

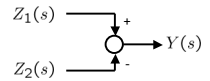


$$\Rightarrow \quad \frac{Y(s)}{R(s)} = G_1(s)G_2(s)$$



Elementary TF block diagrams

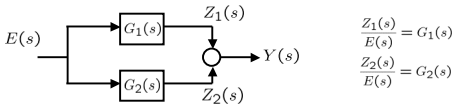
- Summing Junction



$$\Rightarrow \quad Y(s) = Z_1(s) - Z_2(s)$$

Elementary TF block diagrams

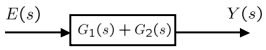
- Parallel connection



$$\frac{Z_1(s)}{E(s)} = G_1(s) \qquad \frac{Z_2(s)}{E(s)} = G_2(s)$$

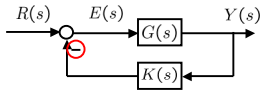
$$Y(s) = Z_1(s) + Z_2(s) = (G_1(s) + G_2(s))E(s)$$

$$\Rightarrow \quad \frac{Y(s)}{E(s)} = G_1(s) + G_2(s)$$



Feedback connection

- Negative feedback system

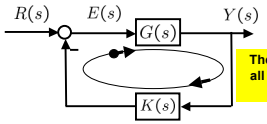


Be careful when computing transfer functions from outside to inside the feedback!

$$E(s) = R(s) - K(s)G(s)E(s) \quad \Rightarrow \quad \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)K(s)}$$

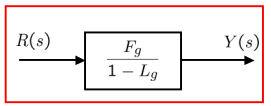
$$Y(s) = G(s)E(s) \quad \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)K(s)}$$

Feedback loop formula, $TF_{R!Y}$



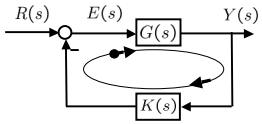
The loop gain is the product of all transfer functions that form the loop

- F_g: Forward gain from R(s) to Y(s) ➡ G(s)
- L_g: Loop gain: ➡ G(s)K(s)(-1)

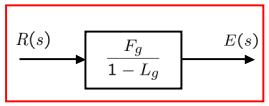


$$\Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)K(s)}$$

Feedback loop formula, $TF_{R!E}$

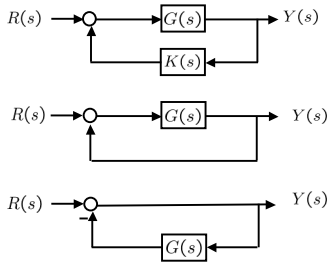


- F_g: Forward gain from R(s) to E(s) ➡ 1
- L_g: Loop gain: ➡ G(s)K(s)(-1)



$$\Rightarrow \quad \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)K(s)}$$

Exercises



Summary and Exercises

- Block diagrams
  - Elementary diagrams
  - Feedback connections
- Next
  - Mechanical Systems
- Exercises
  - Obtain TFs from R to Y on the previous page.
  - Obtain TFs from input F to outputs x_1 and x_2 for the quarter car problem in terms of G_1, G_2, and G_3.

Solving for System Models

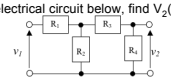
- Using linear algebra to find system models
  - A method to organize your
    - Problem statement
    - Problem equations
    - Problem model solution
- Without an organized approach
  - You get lost in the equations
- We will use electrical networks as an example
  - But… this approach works for all models

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The Problem Statement

- On the top of your paper, rewrite the problem and start the solution by drawing a line and writing "solution:"

2.2 For the electrical circuit below, find $V_2(s)/ V_1(s)$

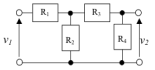


Solution:

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Define the Input and Output

2.2 For the electrical circuit below, find $V_2(s)/ V_1(s)$

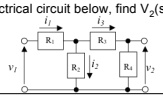


Solution:
Input: v_1
Output: v_2

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Define internal variables

2.2 For the electrical circuit below, find $V_2(s)/ V_1(s)$

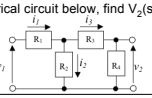


Solution:
Input: v_1
Output: v_2
Variables: i_1, i_2, i_3

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Find enough equations:
n = Output + variable

2.2 For the electrical circuit below, find $V_2(s)/ V_1(s)$



Solution:
Input: v_1
Output: v_2
Variables: i_1, i_2, i_3 => (4) Equations needed

$$v_2 = R_4 i_3$$
$$v_1 - R_1 i_1 - R_2 i_2 = 0$$
$$v_1 - R_1 i_1 - R_3 i_3 - R_4 i_3 = 0$$
$$i_1 - i_2 - i_3 = 0$$

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Rearrange equations:
LHS:Output + Variables, RHS:Input

Solution:
Input: v_1
Output: v_2
Variables: i_1, i_2, i_3 => (4) Equations needed

$$v_2 = R_4 i_3$$
$$v_1 - R_1 i_1 - R_2 i_2 = 0$$
$$v_1 - R_1 i_1 - R_3 i_3 - R_4 i_3 = 0$$
$$i_1 - i_2 - i_3 = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & -R_4 \\ 0 & R_1 & R_2 & 0 \\ 0 & R_1 & 0 & (R_3 + R_4) \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_2 \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} v_1$$

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Solve Matrix equations
Usually using Kramer's rule

$$\frac{v_2}{v_1}=\begin{vmatrix} 0 & 0 & 0 & -R_4 \\ 1 & R_1 & R_2 & 0 \\ 1 & R_1 & 0 & (R_3+R_4) \\ 0 & 1 & -1 & -1 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 0 & 0 & -R_4 \\ 0 & R_1 & R_2 & 0 \\ 0 & R_1 & 0 & (R_3+R_4) \\ 0 & 1 & -1 & -1 \end{vmatrix}$$

$$=\left\{(R_4)\begin{vmatrix} 1 & R_1 & R_2 \\ 1 & R_1 & 0 \\ 0 & 1 & -1 \end{vmatrix}\right\} \Bigg/ \left\{(1)\begin{vmatrix} R_1 & R_2 & 0 \\ R_1 & 0 & (R_3+R_4) \\ 1 & -1 & -1 \end{vmatrix}\right\}$$

$$=\left\{(R_4)\left[(1)\begin{vmatrix} R_1 & 0 \\ 1 & -1 \end{vmatrix}+(-1)\begin{vmatrix} R_1 & R_2 \\ 1 & -1 \end{vmatrix}\right]\right\} \Bigg/ \left\{(1)\left[(R_1)\begin{vmatrix} 0 & (R_3+R_4) \\ -1 & -1 \end{vmatrix}-(R_2)\begin{vmatrix} R_1 & (R_3+R_4) \\ 1 & -1 \end{vmatrix}\right]\right\}$$

$$=\left\{(R_4)\left[(1)(-R_1)+(-1)(-R_1-R_2)\right]\right\} \Big/ \left\{(1)\left[(R_1)(R_3+R_4)-(R_2)(-R_1-R_3-R_4)\right]\right\}$$

$$=\left\{(R_4)\left[(R_2)\right]\right\} \Big/ \left\{\left[(R_1R_3+R_1R_4)+(R_1R_2+R_2R_3+R_2R_4)\right]\right\}$$

$$=(R_2R_4)\Big/(R_1R_2+R_1R_3+R_1R_4+R_2R_3+R_2R_4)$$

The Advantage

- No guesswork on the next step in the solution
  - There is an step-by-step approach
- The approach is modular
  - At each step, you only worry about that step
- You can use insight to shorten the process
  - If you spotted that v_2 is only a function i_3
    - You can solve the 1st equation separately
    - Yielding a 3x3 instead of a 4x4 matrix problem
      - Simpler to solve
- You can look at each small step to find errors

Try it - You will like it

- For dynamic problems, take Laplace transform
  - Use impedance $Z_i(s)$ instead of resistance R_i
  - Remember $Z_R(s) = R$, $Z_L(s) = Ls$, and $Z_C(s) = 1/Cs$,
- Then follow the same process to matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & -Z_4(s) \\ 0 & Z_1(s) & Z_2(s) & 0 \\ 0 & Z_1(s) & 0 & (Z_3(s)+Z_4(s)) \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} V_2(s) \\ I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} V_1(s)$$

  - with the same result

$$\Rightarrow \frac{V_2(s)}{V_1(s)}=\frac{(Z_2Z_4)}{(Z_1Z_2+Z_1Z_3+Z_1Z_4+Z_2Z_3+Z_2Z_4)}$$