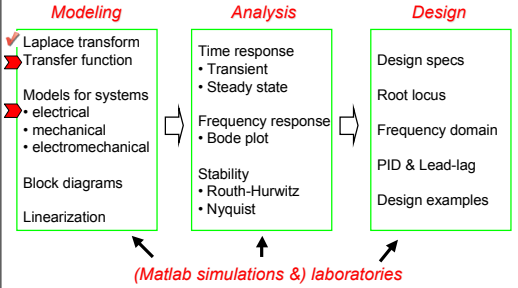
ME451: Control Systems

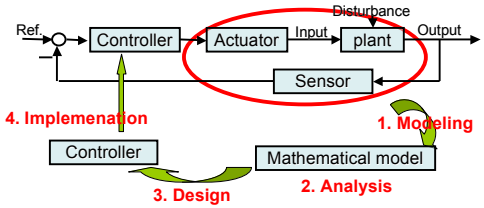
Lecture 4
Modeling of electrical systems

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Department of Mechanical Engineering
Michigan State University

Course roadmap



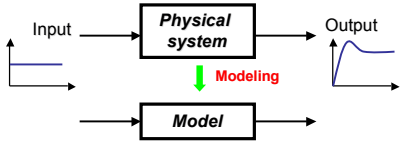
Controller design procedure (review)



- What is the "mathematical model"?
- Transfer function
- Modeling of electrical circuits

Mathematical model

- Representation of the input-output (signal) relation of a physical system



- A model is used for the **analysis** and **design** of control systems.

Important remarks on models

- Modeling is the most important and difficult task in control system design.
- No mathematical model exactly represents a physical system.

$$\text{Math model} \neq \text{Physical system}$$
$$\text{Math model} \approx \text{Physical system}$$

- Do not confuse models with physical systems!
- In this course, we may use the term "system" to mean a mathematical model.

Transfer function

- Defined by

$$G(s) := \frac{Y(s)}{U(s)}$$

Laplace transform of system output
Laplace transform of system input

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

- Zero initial condition are assumed because
  - We are interested in the response we can design
  - IC response disappears with time
  - Final system response depends only on inputs

Impulse response

- Suppose that u(t) is the unit impulse function and system is at rest.

$u(t) = \delta(t)$ → System → $g(t)$

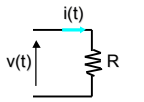
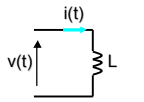
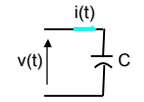
$U(s) = 1$

- The output g(t) for the unit impulse input is called ***impulse response***.
- When U(s)=1, the transfer function can also be defined as the Laplace transform of impulse response:

$$G(s) := \mathcal{L}\{g(t)\}$$

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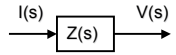
Models of electrical elements: (constitutive equations)

Resistance	Inductance	Capacitance
		
$v(t) = Ri(t)$	$v(t) = L\frac{di(t)}{dt}$	$i(t) = C\frac{dv(t)}{dt}$
Laplace transform	$(i(0) = 0)$	$(v(0) = 0)$
$\frac{V(s)}{I(s)} = R$	$\frac{V(s)}{I(s)} = sL$	$\frac{V(s)}{I(s)} = \frac{1}{sC}$

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Impedance

- Generalized resistance to a sinusoidal alternating current (AC) current I(s)
- Z(s): V(s)=Z(s)I(s)



Element	Time domain	Impedance Z(s)
Resistance	$v(t) = Ri(t)$	$\frac{V(s)}{I(s)} = R$
Inductance	$v(t) = L\frac{di(t)}{dt}$	$\frac{V(s)}{I(s)} = sL$
Capacitance	$i(t) = C\frac{dv(t)}{dt}$	$\frac{V(s)}{I(s)} = \frac{1}{sC}$

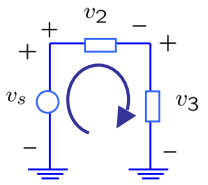
Memorize!

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Kirchhoff's Voltage Law (KVL)

- The algebraic sum of voltage drops around any loop is =0.

$$v_s - v_2 - v_3 = 0$$

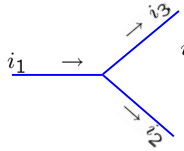


$$v_s = v_2 + v_3$$

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Kirchhoff's Current Law (KCL)

- The algebraic sum of currents into any junction is zero.

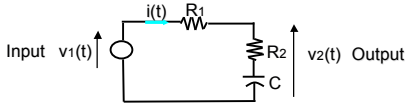


$$i_1 - i_2 - i_3 = 0$$

$$i_1 = i_2 + i_3$$

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Modeling example



- Kirchhoff voltage law (with zero initial conditions)

$$v_1(t) = (R_1 + R_2)i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau$$
$$v_2(t) = R_2 i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau$$

- By Laplace transform, (easiest to do directly)

$$V_1(s) = (R_1 + R_2)I(s) + \frac{1}{sC}I(s)$$
$$V_2(s) = R_2 I(s) + \frac{1}{sC}I(s)$$

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Modeling example (cont'd)

- By Laplace transform, (easiest to do directly)

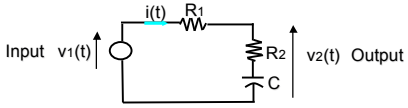
$$V_1(s) = (R_1 + R_2)I(s) + \frac{1}{sC}I(s)$$
$$V_2(s) = R_2 I(s) + \frac{1}{sC}I(s)$$

- Transfer function

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{sC}}{(R_1+R_2) + \frac{1}{sC}} = \frac{R_2 Cs + 1}{(R_1+R_2)Cs + 1}$$

(first-order system)

Modeling example (cont'd)



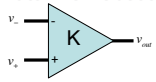
- Transfer function

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{sC}}{(R_1+R_2) + \frac{1}{sC}} = \frac{R_2 Cs + 1}{(R_1+R_2)Cs + 1}$$

(first-order system)

Example: Modeling of op amp

- Op Amps: The Amplifier you want to know about
  - Op Amp Behavior is simple



  - Voltage:

$v_{out} = K(v_+ - v_-)$ where $K \to 10^6 \approx \infty$

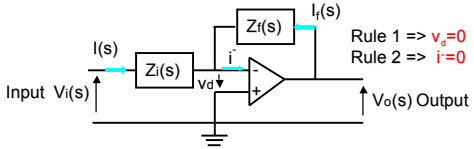
means for finite v_{out}, $\boxed{v_+ = v_-}$

  - Current:

$i_+ = i_- = v_-/R$ where $R \to 10^6$

means $\boxed{i_+ = i_- = 0}$

Example: Modeling of op amp



- Impedance Z(s): V(s)=Z(s)I(s)

$$(V_o(s) - 0) = Z_f(s)I_f(s)$$
$$(V_i(s) - 0) = Z_i(s)I(s)$$
$$0 = I(s) + I_f(s)$$

Example: Modeling of op amp

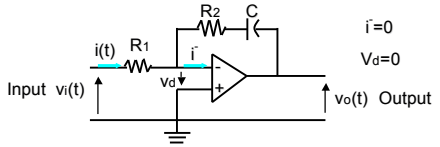
- Impedance Z(s): V(s)=Z(s)I(s)

$$(V_o(s) - 0) = Z_f(s)I_f(s)$$
$$(V_i(s) - 0) = Z_i(s)I(s)$$
$$0 = I(s) + I_f(s)$$

- Transfer function of the above op amp:

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_f(s)(I_f(s) = -I(s))}{Z_i(s)I(s)} = -\frac{Z_f(s)}{Z_i(s)}$$

Modeling example: op amp

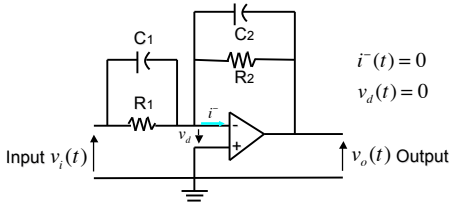


- By the formula in previous two pages,

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{-(R_2 + \frac{1}{sC})}{R_1} = -\frac{R_2 Cs + 1}{R_1 Cs}$$

(first-order system)

Modeling exercise: op amp

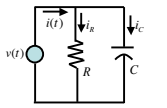


$$i^-(t) = 0$$
$$v_d(t) = 0$$

Find the transfer function $G(s) = \dfrac{v_o(t)}{v_i(t)}$

Modeling exercise: op amp

- We need the impedance of the parallel RC circuit



$$V(s) = (R) I_R(s)$$
$$V(s) = \left(\frac{1}{Cs}\right) I_C(s)$$
$$I(s) = I_R(s) + I_C(s)$$

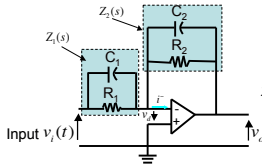
- Find the impedance transfer function

$$I(s) = I_R(s) + I_C(s) = \left(\frac{V(s)}{R}\right) + (CsV(s)) = \left(\frac{1}{R} + Cs\right) V(s)$$
$$= \left(\frac{1 + RCs}{R}\right) V(s)$$

$$Z(s) = \frac{V(s)}{I(s)} = \left(\frac{R}{1 + RCs}\right)$$

Modeling exercise: op amp

- With the impedence, we can now model the op amp circuit



$$Z_2(s) = \left(\frac{R_2}{1 + R_2 C_2 s}\right)$$
$$Z_1(s) = \left(\frac{R_1}{1 + R_1 C_1 s}\right)$$

- The Transfer Function

$$G(s) = \frac{V_o(s)}{V_o(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2(1 + R_1 C_1 s)}{R_1(1 + R_2 C_2 s)}$$

Summary & Exercises

- Modeling
  - Modeling is an important task!
  - Mathematical model
  - Transfer function
  - Modeling of electrical systems
- Next, modeling of mechanical systems
- Exercises
  - Read Sections 2.2, 2.3
  - Solve problems 2.1, 2.2, and 2.3 in page 62