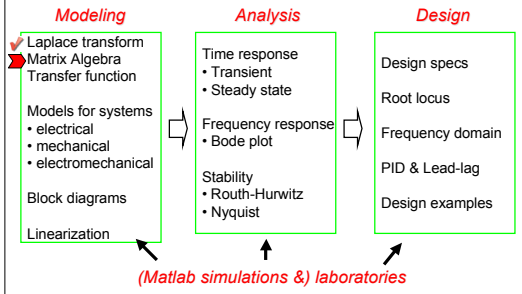
ME451: Control Systems

Lecture 3
The Matrix Inverse

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Course roadmap



Square Matrices have Inverses

- For a Matrix **A,**
  - $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$**, the Identity Matrix**
    - So "$\mathbf{A}^{-1}$" plays the role of dividing by **A**
    - There is no "divide" operation for matrices
- For the matrix **A**, the matrix equation $\mathbf{Ay} = \mathbf{f}$
  - Has the solution $\mathbf{y} = \mathbf{A}^{-1}\mathbf{f}$ if is "non-singular"
- How do we compute $\mathbf{A}^{-1}$?

The Minor of a Matrix

- The *minor* m_{ij} for element a_{ij} of a square matrix $\mathbf{A}$ is the determinant of $\mathbf{A}$ after deleting the i^{th} row and j^{th} column of $\mathbf{A}$.
- Example:
  For $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix}$
- The minor $m_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$

The Cofactor

- The *cofactor* c_{ij} of an element a_{ij} of a square matrix $\mathbf{A}$ is

$$c_{ij} = (-1)^{i+j} m_{ij} \qquad \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(the "signed minor") (the "sign"pattern)

- In the preceding example:

$$c_{31} = (-1)^{3+1}(m_{31}) = (-1)^4(-2) = -2$$

The Adjoint

- The *adjoint* of a square matrix $\mathbf{A}$ is the transpose of it's matrix of cofactors. For the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

in our example, the adjoint of $\mathbf{A}$ is

$$\mathrm{Adj}\mathbf{A} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} 4 & -5 & -1 \\ -2 & 4 & 0 \\ -2 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 & -2 \\ -5 & 4 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

Matrix Inverse

- The inverse of a square matrix $\mathbf{A}$ denoted by $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1} = \frac{\text{Adj}\,\mathbf{A}}{|\mathbf{A}|}$$

- For our example,

$$\mathbf{A}^{-1} = \frac{\text{Adj}\,\mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 4 & -2 & -2 \\ -5 & 4 & 3 \\ -1 & 0 & 1 \end{bmatrix} \Big/ \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 3 \end{vmatrix} =$$

$$|\mathbf{A}| = (2)\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (2)(3+1) - (3+2) + (1-2) = 2$$

$$\mathbf{A}^{-1} = \frac{\text{Adj}\,\mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 2 & -1 & -1 \\ -2.5 & 2 & 1.5 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

7

Test the Inverse

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -2.5 & 2 & 1.5 \\ -0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (4-1-2) & (2-1-1) & (2+1-3) \\ (-5+2+3) & (-2.5+2+1.5) & (-2.5-2+4.5) \\ (-1+0+1) & (-0.5+0+0.5) & (-0.5+0+1.5) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

It Works !!!

8

Summary & Exercises

- The Adjoint Method produces a correct matrix inverse
  - But it uses determinants - computationally "big"
  - But The Adjoint is easier for humans
    - Gaussian Elimination is easier for computers….
- Test this with Matlab
- Work on quizzes and
- Test quiz results with Matlab

9

Summary and Exercises

10