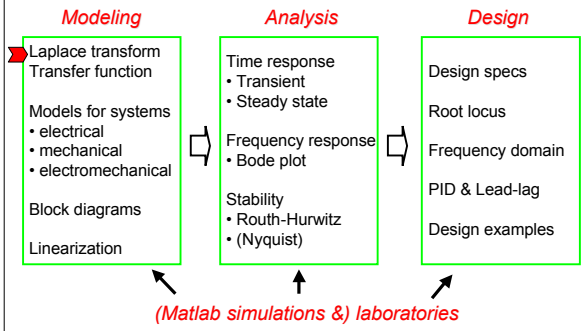
ME451: Control Systems

Lecture 2
Laplace transform

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Course roadmap

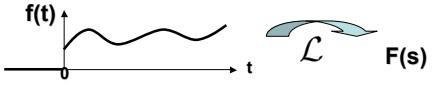


Laplace transform

- One of most important math tools in the course!
- Definition: For a function f(t) (f(t)=0 for t<0),

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt$$

(s: complex variable)

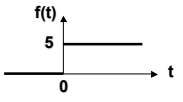


- We denote Laplace transform of f(t) by F(s).

Example of Laplace transform

- Step function

$$f(t) = 5u(t) = \begin{cases} 5 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty 5e^{-st}\,dt = 5\int_0^\infty e^{-st}\,dt$$

$$= 5\left[-\frac{1}{s}\left[e^{-st}\right]_0^\infty\right] = 5\left[\frac{1}{s}\right] = \frac{5}{s}$$

Remember ℒ(u(t)) = 1/s

Integration is Hard

Tables are Easier

Laplace transform table

(Table B.1 in Appendix B of the textbook)

	Time Function $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$	
1	1	$\frac{1}{s}$	$s > 0$
2	t (unit-ramp function)	$\frac{1}{s^2}$	$s > 0$
3	t^n (n, a positive integer)	$\frac{n!}{s^{n+1}}$	$s > 0$
4	e^{at}	$\frac{1}{s-a}$	$s > a$
5	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
6	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$s > 0$
7	$t^n g(t)$, for $n = 1, 2, \ldots$	$(-1)^n \frac{d^n G(s)}{ds^n}$	
8	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$s > \lvert\omega\rvert$
9	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$s > \lvert\omega\rvert$
10	$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$	Scale property
11	$e^{at}g(t)$	$G(s-a)$	Shift property
12	$e^{at}t^n$, for $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$

ℒ

ℒ⁻¹

Inverse Laplace Transform

Properties of Laplace transform

Linearity

$$\mathcal{L}\{\alpha_1 f_1(t)+\alpha_2 f_2(t)\}=\alpha_1 F_1(s)+\alpha_2 F_2(s)$$

Proof. $\mathcal{L}\{\alpha_1 f_1(t)+\alpha_2 f_2(t)\} = \int_0^\infty (\alpha_1 f_1(t)+\alpha_2 f_2(t))e^{-st}dt$

$= \alpha_1 \underbrace{\int_0^\infty f_1(t)e^{-st}dt}_{F_1(s)} + \alpha_2 \underbrace{\int_0^\infty f_2(t)e^{-st}dt}_{F_2(s)}$

Ex.

$$\mathcal{L}\{5u_s(t)+3e^{-2t}\}=5\mathcal{L}\{u_s(t)\}+3\mathcal{L}\{e^{-2t}\}=\frac{5}{s}+\frac{3}{s+2}$$

Properties of Laplace transform

Differentiation

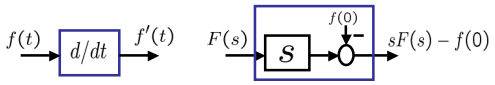
$$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$$

Proof.

$$\mathcal{L}\{f'(t)\}=\int_0^\infty f'(t)e^{-st}dt=\left[f(t)e^{-st}\right]_0^\infty+s\int_0^\infty f(t)e^{-st}dt=sF(s)-f(0)$$

Ex. $\mathcal{L}\{(\cos 2t)'\}=s\mathcal{L}\{\cos 2t\}-1=\frac{s^2}{s^2+4}-1=\frac{-4}{s^2+4}(=\mathcal{L}\{-2\sin 2t\})$

t-domain s-domain



Properties of Laplace transform

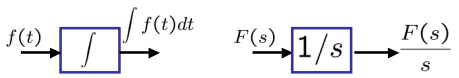
Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\frac{F(s)}{s}$$

Proof. $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \left(\int_0^t f(\tau)d\tau\right)e^{-st}dt$

$= -\frac{1}{s}\left[\left(\int_0^t f(\tau)d\tau\right)e^{-st}\right]_0^\infty+\frac{1}{s}\int_0^\infty f(t)e^{-st}dt=\frac{F(s)}{s}$

t-domain s-domain



Properties of Laplace transform

Final value theorem

$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$ if all the poles of sF(s) are in the left half plane (LHP)

Ex. $F(s)=\frac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to\infty} f(t)=\lim_{s\to 0}\frac{5}{s^2+s+2}=\frac{5}{2}$

Poles of sF(s) are in LHP, so final value thm applies.

Ex. $F(s)=\frac{4}{s^2+4} \Rightarrow \lim_{t\to\infty} f(t) \not\exists \lim_{s\to 0}\frac{4s}{s^2+4}=0$

Some poles of sF(s) are not in LHP, so final value thm does NOT apply.

Properties of Laplace transform

Initial value theorem

$\lim_{t\to 0+} f(t)=\lim_{s\to\infty} sF(s)$ if the limits exist.

Remark: In this theorem, it does not matter if pole location is in LHS or not.

Ex. $F(s)=\frac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to 0+} f(t)=\lim_{s\to\infty} sF(s)=0$

Ex. $F(s)=\frac{4}{s^2+4} \Rightarrow \lim_{t\to 0+} f(t)=\lim_{s\to\infty} sF(s)=0$

Properties of Laplace transform

Convolution

$F_1(s)=\mathcal{L}\{f_1(t)\}$
$F_2(s)=\mathcal{L}\{f_2(t)\}$

Convolution

$F_1(s)F_2(s) = \mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right\}$

$\Rightarrow \quad = \mathcal{L}\left\{\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right\}$

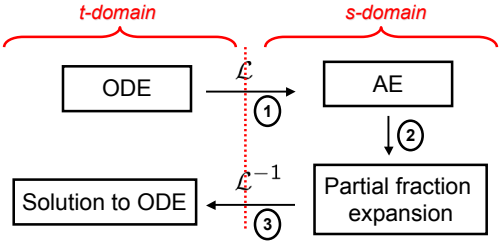
IMPORTANT REMARK

$$F_1(s)F_2(s) \not\exists \mathcal{L}\{f_1(t)f_2(t)\}$$

$$L^{-1}\left(F_1(s)F_2(s)\right)\neq f_1(t)f_2(t)$$

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Example 1

- 1st Order ODE with input and Initial Condition

$$5\dot{y}(t)+10y(t)=3u(t) \qquad y(0)=1$$

- Take Laplace Transform

$$5\left[sY(s)-y(0)\right]+10\left[Y(s)\right]=3\left[\frac{1}{s}\right]$$

- Solve for $Y(s)$

$$(5s+10)Y(s)=5y(0)+3\left[\frac{1}{s}\right]$$

$$Y(s)=\frac{5}{(5s+10)}+\frac{3}{s(5s+10)}=\frac{1}{(s+2)}+\frac{0.6}{s(s+2)}$$

(Initial Condition) + (Input)

Example 1 (cont)

- Use table to Invert $Y(s)$ term by term to find y(t)

$$Y(s)=\frac{1}{(s+2)}+\frac{0.6}{s(s+2)}$$

- From the Table:

$$L\left(\frac{1}{s+a}\right)=e^{-at} \quad \Rightarrow \quad L^{-1}\left(\frac{1}{s+2}\right)=e^{-2t}$$

$$L\left(\frac{a}{s(s+a)}\right)=\left(1-e^{-at}\right) \quad \Rightarrow \quad L^{-1}\left(\frac{(0.3)2}{s(s+2)}\right)=0.3\left(1-e^{-2t}\right)$$

- So that

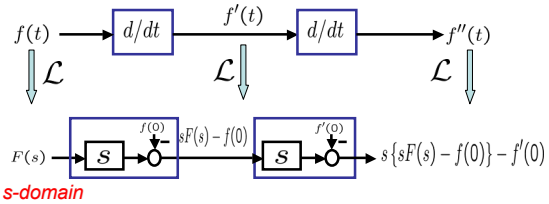
$$y(t)=e^{-2t}+0.3\left(1-e^{-2t}\right)$$

(Initial Condition) + (Input)

Properties of Laplace transform
Differentiation (review)

$$\mathcal{L}\left\{f'(t)\right\}=sF(s)-f(0)$$

t-domain



s-domain

Example 2

$$\ddot{y}(t)-y(t)=t, \quad y(0)=1, \dot{y}(0)=1$$

$$s^2Y(s)-sy(0)-\dot{y}(0)-Y(s)=\frac{1}{s^2},$$

$$s^2Y(s)-s-1-Y(s)=\frac{1}{s^2} \Rightarrow \left(s^2-1\right)Y(s)=(s+1)+\frac{1}{s^2}$$

$$Y(s)=\frac{(s+1)}{\left(s^2-1\right)}+\frac{1}{s^2\left(s^2-1\right)}$$

$$Y(s)=\frac{A}{(s-1)}+\frac{B}{(s+1)}+\frac{C}{s^2} \quad \text{(Find A, B and C)}$$

$$Y(s)=\frac{(3/2)}{(s-1)}+\frac{(-1/2)}{(s+1)}+\frac{(-1)}{s^2}$$

$$y(t)=(3/2)e^{t}+(-1/2)e^{-t}+(-1)t$$

How do we do that???

Example 2 (cont'd)

Partial fraction expansion

unknowns

$$Y(s)=\frac{(s+1)}{\left(s^2-1\right)}+\frac{1}{s^2\left(s^2-1\right)}=\frac{A}{(s-1)}+\frac{B}{(s+1)}+\frac{C}{s^2}$$

Multiply both sides by (s-1) & let s ⇨1

$$(s-1)Y(s)\big|_{s\to 1}=(s-1)\left[\frac{A}{(s-1)}+\frac{B}{(s+1)}+\frac{C}{s^2}\right]=A$$

$$so \quad A=(s-1)\left\{\frac{(s+1)}{\left(s^2-1\right)}+\frac{1}{s^2\left(s^2-1\right)}\right\}\Big|_{s\to 1}=\left\{1+\frac{1}{s^2(s+1)}\right\}=1+\frac{1}{2}=\frac{3}{2}$$

Similarly,

$$B=(s+1)Y(s)\big|_{s\to -1}=-\frac{1}{2}$$

$$C=\left(s^2\right)Y(s)\big|_{s\to 0}=-1$$

Example 3

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u_s(t),\ y(0)=-1,\ y'(0)=2$$

Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\{y''(t)\}}+3\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\{y'(t)\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow\ Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$ (This also isn't in the table…)

Example 3 (cont'd)

Inverse Laplace transform

$$\mathcal{L}^{-1}\left\{Y(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}\right\}$$

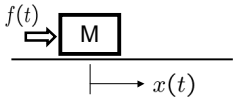
$$\Rightarrow\ y(t)=\left(\underbrace{\frac{5}{2}}_{A}+\underbrace{(-5)}_{B}e^{-t}+\underbrace{\frac{3}{2}}_{C}e^{-2t}\right)u_s(t)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t\to\infty}y(t)=\lim_{s\to0}sY(s)=\lim_{s\to0}\frac{-s^2-s+5}{(s+1)(s+2)}=\frac{5}{2}$$

Example: Newton's law

$$M\frac{d^2x(t)}{dt^2}=f(t)$$



We want to know the trajectory of x(t). By Laplace transform,

$$M\left(s^2X(s)-sx(0)-x'(0)\right)=F(s)$$

$$\Rightarrow\ X(s)=\underbrace{\frac{1}{Ms^2}F(s)}+\underbrace{\frac{x(0)}{s}+\frac{x'(0)}{s^2}}$$

(Total response) = (Forced response) + (Initial condition response)

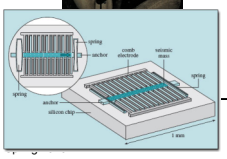
$$\Rightarrow\ x(t)=\mathcal{L}^{-1}\left[\frac{1}{Ms^2}F(s)\right]+x(0)u_s(t)+x'(0)tu_s(t)$$

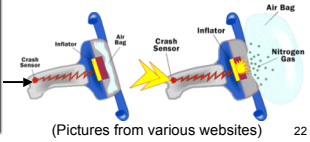
EX. Air bag and accelerometer

- Tiny MEMS accelerometer
  - Microelectromechanical systems (MEMS)









(Pictures from various websites)

In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!

Summary & Exercises

- Laplace transform (Important math tool!)
  - Definition
  - Laplace transform table
  - Properties of Laplace transform
  - Solution to ODEs via Laplace transform
- Exercises
  - Read Appendix A, B.
  - Solve Quiz problems…