ME451: Control Systems

Lecture 1
Math Review 1

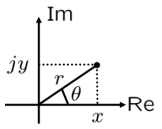
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Prerequisites: Complex Numbers

- Ordered pair of two real numbers
  $s := x + jy \qquad j = \sqrt{-1}$
- Conjugate
  $\bar{s} = s^* := x - jy$
- Addition $s_1 = x_1 + jy_1, s_2 = x_2 + jy_2$
  $s_1 + s_2 = (x_1 + x_2) + j(y_1 + y_2)$
- Multiplication
  $s_1 s_2 = (x_1 + jy_1)(x_2 + jy_2)$
  $= (x_1 x_2 - y_1 y_2) + j(y_1 x_2 + x_1 y_2)$

Complex Numbers: Polar vs. Rectangular Form

- **Euler's identity** $e^{j\theta} := \cos\theta + j\sin\theta$
- **Polar form** $s := x + jy = re^{j\theta}$
- **Magnitude** $r = \sqrt{x^2 + y^2}$
- **Phase** $\theta = \tan^{-1}(y/x)$



Complex Numbers: Polar Arithmetic

- For two polar complex numbers…
  $s_1 = r_1 e^{j\theta_1},\ s_2 = r_2 e^{j\theta_2}$
- Multiplication is easy
  $s_1 s_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
- Division is also
  $\frac{s_1}{s_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
- Addition and subtraction are a mess…
  - Convert to rectangular, add then convert back

Matrices

- An nxm array of numbers is a matrix
  - nx1 arrays are vectors (a column)

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,1} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & & \\ \vdots & & \ddots & \\ a_{n,1} & \cdots & & a_{n,m} \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\mathbf{a}^T = \mathbf{a}' = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

  - 1xn arrays are transposed vectors

Matrix Multiply

- Multiply rows by columns

$$\mathbf{AB} = [a_{i,j}][b_{i,j}] = \underset{(2x3)}{\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}} \underset{(3x2)}{\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix}}$$

$$= \underset{(2x2)}{\begin{bmatrix} (a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1}) & (a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2}) \\ (a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1}) & (a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2}) \end{bmatrix}}$$

- There is no matrix divide, only inverses

Matrix Determinant

- Pick any row or column…
  - The determinant is the sum of the "signed row elements" times the determinants of there "minors"
  - The i,j minor is the matrix remaining after the i^{th} row and j^{th} column are deleted from the matrix

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 1[4(1) - 3(1)] - 2[3(1) - 0(1)] + 0[\ ]$$

$$= 1 - 6 + 0 = -5$$

  - There is NO OTHER WAY!!!! (No Short cuts!!!)

Kramer's Rule

- For the nxn linear equation $\mathbf{Ay = f}$
- The solution is $y_j = \dfrac{|A^*|}{|A|}$
  - Where $\mathbf{A^*}$ is the matrix $\mathbf{A}$ with $\mathbf{f}$ substituted for the j^{th} column

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y_1 = \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = (2-6)/2 = -2$$

A Little Matlab

Comment	Matlab
>> Define the matrix A (A is echoed)	>> A=[1 3;0 2] A = 1 3 0 2
>> Define the vector f (f is echoed)	>> f=[1;2] f = 1 2
>> compute y=(inv(A))*f y is echoed with y_1 = -2 and y_2 = 1	>> y=A\f y = -2 1

Logarithm

- Definition

$$a = b^{\log_b(a)}$$

$$a = 10^{\log_{10}(a)}$$

$$a = e^{\ln(a)} = e^{\log_e(a)}$$

- Use

$$\log(ab) = \log(a) + \log(b)$$

Next Time …

- ***Laplace Transforms***
  - ODE's become Linear Algebraic Equations
- Read the Appendices
- Take a look at the Old Math Quizzes
- Work on "2004B" Math Quiz homework