ME451: Control Systems

Lecture 13
Typical System Design Problems

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Course roadmap

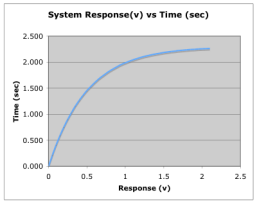
Modeling	*Analysis*	*Design*
✔ Laplace transform	✔ Time response	Design specs
✔ Transfer function	✔ Transient	Root locus
✔ Models for systems	✔ Steady state	Frequency domain
✔ electrical	Frequency response	PID & Lead-lag
✔ mechanical	• Bode plot	Design examples
✔ electromechanical	Stability	
✔ Block diagrams	• Routh-Hurwitz	
✔ Linearization	• Nyquist	

(Matlab simulations &) laboratories

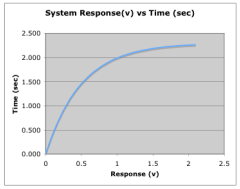
Typical Problems using Design Specifications

System Models

- Given a system step response, find a system model
- What's the approach?



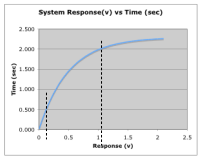
System Models



1. What is the format of the model? $\frac{C(s)}{R(s)} = G(s) = \frac{k}{\tau s + 1}$
2. What are the Parameters?
   Estimate…

$k = 2.3$ (look at graph) $\tau = 0.5$ (63%*2.3=1.45, @c(t)=1.45, t=tau=0.5)

System Models



$$\frac{C(s)}{R(s)} = G(s) = \frac{k}{\tau s + 1}$$

$$c(t) = k\left(1 - e^{-t/\tau}\right)$$

Iterative solution

tau=	0.5387
x=exp(-1/tau)=	0.1563
0.5*(1-x^1.05)-2.0*(1-x^0.13) =	9.5693E-07
k=0.5/(1-x^0.13)=	2.33215457

Calculate the Parameters…get (2) points (c,t) = (0.5, 0.13), (2.0,1.05)

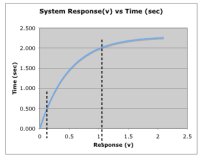
$$\left.\begin{aligned} 0.5 &= k(1 - e^{-0.13/\tau}) \\ 2.0 &= k(1 - e^{-1.05/\tau}) \end{aligned}\right\} \Rightarrow \frac{0.5}{(1 - e^{-0.13/\tau})} = k = \frac{2.0}{(1 - e^{-1.05/\tau})}$$

for $x = e^{-1/\tau}$, $0.5\left(1 - x^{1.05}\right) = 2.0\left(1 - x^{0.13}\right)$

or $0.5\left(1 - x^{1.05}\right) - 2.0\left(1 - x^{0.13}\right) = 0$

Then iterate…
(nonlinear equations are a pain)

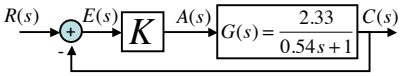
System Models



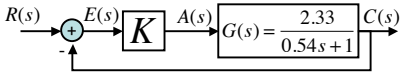
The Model:

$$G(s) = \frac{2.33}{0.54s+1}$$

Now let's close the loop



Closed-loop System Model



The Closed-Loop Model:

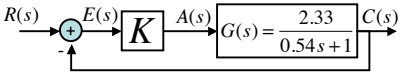
$$E(s) = R(s) - \left(\frac{2.33}{0.54s+1}\right)KE(s)$$

$$\left[1+\left(\frac{2.33K}{0.54s+1}\right)\right]E(s) = R(s)$$

$$E(s) = \frac{R(s)}{\left[1+\left(\frac{2.33K}{0.54s+1}\right)\right]} \quad \text{but } C(s) = \left(\frac{2.33K}{0.54s+1}\right)E(s)$$

$$\Rightarrow C(s) = \frac{2.33K}{\left[(0.54s+1)+2.33K\right]}R(s) \Rightarrow G_{CL} = \frac{C(s)}{R(s)} = \frac{\left(2.33K/(1+2.33K)\right)}{\left(0.54/(1+2.33K)\right)s+1}$$

Closed-Loop System Model



The Closed-Loop Model:

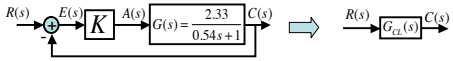
$$G_{CL} = \frac{C(s)}{R(s)} = \frac{k}{\tau s+1} = \frac{\left(2.33K/(1+2.33K)\right)}{\left(0.54/(1+2.33K)\right)s+1}$$

As $K \to \infty$ (BIG),

Closed-Loop Gain $k \to 1$ $(C(s) \approx R(s))$

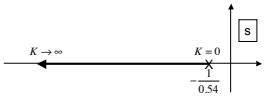
Closed-Loop Time Constant $\tau \to 0$ (FASTER)

Closed-Loop System Model



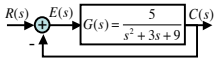
The Closed-Loop Model:

$$G_{CL}(s) = \frac{C(s)}{R(s)} = \frac{\left(2.33K/(1+2.33K)\right)}{\left(0.54/(1+2.33K)\right)s+1}$$



$G_{cl}(s)$ is always "negative real" (Stable, No Oscillation) with increasing control gain K

Open vs Closed Loop Response



a) Sketch the Open-Loop and Closed-Loop step responses
b) Compute the OL and CL step responses with Matlab

Open-Loop Standard Form: $G(s) = \dfrac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

2nd order OL parameters:

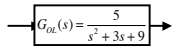
$$\left.\begin{matrix} k\omega_n^2 = 5 \\ 2\zeta\omega_n = 3 \\ \omega_n^2 = 9 \end{matrix}\right\} \begin{matrix} k = 5/9 = 0.56 \\ \zeta = 1/2 = 0.50 \\ \omega_n = 3 \end{matrix}$$

steady-state (DC) gain = 0.56

time constant, $\tau = 1/\zeta\omega_n = 0.67$ sec

damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.6$ rad/sec

Open Loop Response

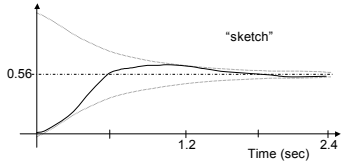


2nd order open-loop parameters:

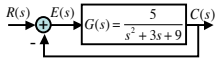
steady-state (DC) gain = 0.56, time constant, $\tau = 1/\zeta\omega_n = 0.67$ sec

frequency $\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.6$ rad/sec = $(2.6/2\pi)Hz = 0.41Hz$

(period=1/0.41=2.42sec $\approx 2.42/0.67 = 3.6$ time constants



Closed-Loop Response



Closed-Loop TF: $C(s) = G(s)E(s)$, $E(s) = R(s) - G(s)E(s) \Rightarrow E(s) = R(s)/(1+G(s))$
$C(s) = [G(s)/(1+G(s))]R(s)$

$$G_{CL}(s) = \frac{C(s)}{R(s)} = \frac{G_{OL}(s)}{1+G_{OL}(s)} = \frac{\left(5/\left(s^2+3s+9\right)\right)}{1+\left(5/\left(s^2+3s+9\right)\right)} = \frac{5}{s^2+3s+9+5} = \frac{5}{s^2+3s+14}$$

2nd order CL parameters:

$k\omega_n^2 = 5$, $2\zeta\omega_n = 3$, $\omega_n^2 = 14$

$\omega_n = \sqrt{14} = 3.74$, $k = 5/14 = 0.36$, $\zeta = 3/(2*3.74) = 0.40$

steady-state (DC) gain = 0.36

time constant, $\tau = 1/\zeta\omega_n = 0.67$ sec

damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.43$ rad/sec

Closed-Loop Response

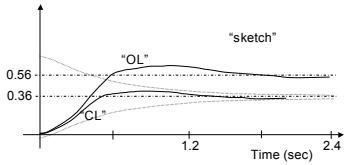
$$G_{CL}(s) = \frac{5}{s^2+3s+14}$$

2nd order open-loop parameters:

steady-state (DC) gain = 0.36, time constant, $\tau = 1/\zeta\omega_n = 0.67$ sec

frequency $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.43$ rad/sec = $\left(3.43/2\pi\right)Hz = 0.55Hz$

(period=1/0.55 = 1.83 sec ≈ 1.83/0.67 = 2.75 time constants



Matlab Simulation

- Open-Loop vs Closed-loop

$$G_{OL}(s) = \frac{5}{s^2+3s+9} \qquad G_{CL}(s) = \frac{5}{s^2+3s+14}$$

Parameter	Open-Loop	Closed-Loop
Gain, k	0.56	0.36
Damping Ratio, ζ	0.50	0.40
Nat. Freq., ω_n	3.0 (rad/sec)	3.74 (rad/sec)
Time Constant, τ	0.67	0.67
Damped Nat Freq, ω_d	2.6 (rad/sec)	3.43 (rad/sec)

Matlab Simulation

- **Open-Loop vs Closed-loop**

$$G_{OL}(s) = \frac{5}{s^2+3s+9} \qquad G_{CL}(s) = \frac{5}{s^2+3s+14}$$

- Matlab Commands

Use:

"step(num,den,tfinal)" to plot, and

"hold on/off" to hold 1st plot and release plot at end

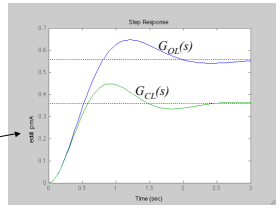
>> step([5],[1 3 9],3); hold on; step([5],[1 3 14],3); hold off

Matlab Simulation

- **Open-Loop vs Closed-loop**

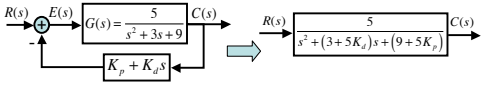
$$G_{OL}(s) = \frac{5}{s^2+3s+9} \qquad G_{CL}(s) = \frac{5}{s^2+3s+14}$$

- Matlab Output:



"bug"
Says "Amplitude"

Closed-Loop Design



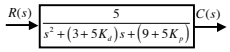
- For the above Closed-Loop System, the TF is

$$G_{CL}(s) = \frac{C(s)}{R(s)} = \frac{\left(\frac{5}{\left(s^2+3s+9\right)}\right)}{\left(1+\left(K_p+K_d s\right)\left(\frac{5}{\left(s^2+3s+9\right)}\right)\right)}$$

$$= \frac{5}{\left(\left(s^2+3s+9\right)+5\left(K_p+K_d s\right)\right)} = \frac{5}{s^2+\left(3+5K_d\right)s+\left(9+5K_p\right)}$$

(You should know how to find this….)

Closed-Loop Design



- For the above Closed-Loop System,
- Find the control gains K_p and K_d so the the specifications
  - 2% settling time < 4 seconds, and
  - overshoot < 20% are achieved.
- The specifications require
  - Time constant τ < 1 second and
  - Damping ratio ζ such that (see equ 4-32 and Fig. 4.8 in text)

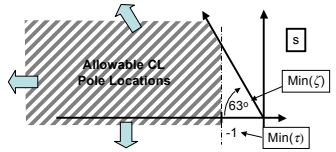
$$20\% = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \Rightarrow \ln(0.2) = -\zeta\pi/\sqrt{1-\zeta^2} \Rightarrow -1.61 = -\zeta\pi/\sqrt{1-\zeta^2}$$

$$\Rightarrow (-1.61)^2(1-\zeta^2) = \zeta^2\pi^2 \Rightarrow 2.59(1-\zeta^2) = \zeta^2\pi^2$$

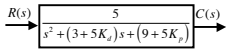
$$\Rightarrow 2.59 = \zeta^2(\pi^2 + 2.59) \Rightarrow \zeta = \sqrt{\frac{2.59}{(\pi^2 + 2.59)}} = 0.46$$

Closed-Loop Design

- For the above Closed-Loop System,
- Find the control gains K_p and K_d so the the specifications
  - 2% settling time < 4 seconds, and
  - overshoot < 20% are achieved.
- The specifications require
  - Time constant τ < 1 second and
  - Damping ratio ζ >0.46 (for ζ = 0.46, cos(θ) = 0.46, θ = 63°)



Closed-Loop Design



For the above Closed-Loop System, we require
Time Constant τ < 1 second and Damping Ratio ζ < 0.46
This is achieved when

$$\tau = \frac{1}{\zeta\omega_n} < 1 \text{ and } \zeta > 0.46 \text{ where}$$

$$\Rightarrow \tau = \frac{1}{\zeta\omega_n} < 1 \quad \Rightarrow \quad \omega_n > \frac{1}{0.46} = 2.2 \text{ (rad/sec)}$$

where

$$(3 + 5K_d) = 2\zeta\omega_n \text{ and } (9 + 5K_p) = \omega_n^2$$

so with $K_p = (\omega_n^2 - 9)/5 \Rightarrow \boxed{K_p > (2.2^2 - 9)/5 = -0.84}$ and

with $K_d = (2\zeta\omega_n - 3)/5 \Rightarrow \boxed{K_d > (2(0.46)(2.2) - 3)/5 = -0.20}$

Summary:

- 1st order and second order models have responses governed by standard sets of parameters
- These standard parameters are defined from…
  - Convenient mathematical model parameters
  - Model "physical response characteristics"
- Systems can be designed by "control" of those standard parameters