ME451: Control Systems

Lecture 11
More Linearization &
1st Order Time Response

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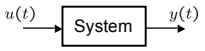
Course roadmap

Modeling	Analysis	Design
✔ Laplace transform	Time response • Transient • Steady state	Design specs
✔ Transfer function		Root locus
✔ Models for systems ✔ electrical ✔ mechanical ✔ electromechanical ✔ Block diagrams Linearization	Frequency response • Bode plot Stability • Routh-Hurwitz • Nyquist	Frequency domain PID & Lead-lag Design examples

(Matlab simulations &) laboratories

What is a linear system?

- A system having ***Principle of Superposition***



$$\left.\begin{array}{l} u_1(t) \to y_1(t) \\ u_2(t) \to y_2(t) \end{array}\right\} \Rightarrow \alpha_1 u_1(t) + \alpha_2 u_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$$
$$\forall \alpha_1, \alpha_2 \in \mathbb{R}$$

A nonlinear system does not satisfy the principle of superposition.

Six Steps to Linearization

1. Identify the system model's inputs and outputs
2. Express the model in the form
   $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) = 0$
3. Find an equilibrium where all input and output derivatives are zero and the original model is satisfied at the operating point (r_o, c_o)
   $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) = f(r_o,0,0,\ldots,c_o,0,0,\ldots) = 0$
   That is …. Solve $f(r_o,c_o)=0$.
   Typically you are given the output c_o and must find the input r_o

Six Steps to Linearization

4. Perform a Taylor Series expansion about the operating point retaining only 1st derivative terms.

$$f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) \cong f(r_o,0,0,\ldots,c_o,0,0,\ldots) + \left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}(r-r_o) + \left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}(\dot{r}-\dot{r}_o) + \left.\frac{\partial f}{\partial \ddot{r}}\right|_{(r_o,c_o)}(\ddot{r}-\ddot{r}_o) + \ldots + \left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}(c-c_o) + \left.\frac{\partial f}{\partial \dot{c}}\right|_{(r_o,c_o)}(\dot{c}-\dot{c}_o) + \left.\frac{\partial f}{\partial \ddot{c}}\right|_{(r_o,c_o)}(\ddot{c}-\ddot{c}_o) + \ldots$$

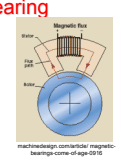
5. Change variables
   $\tilde{r} = (r - r_o), \tilde{\dot{r}} = (\dot{r} - \dot{r}_o), \tilde{\ddot{r}} = (\ddot{r} - \ddot{r}_o), etc.$
   $\tilde{c} = (c - c_o), \tilde{\dot{c}} = (\dot{c} - \dot{c}_o), \tilde{\ddot{c}} = (\ddot{c} - \ddot{c}_o), etc.$
6. Rewrite the Taylor Expansion in the new variables

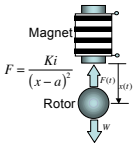
$$\left[\left.\frac{\partial f}{\partial \ddot{r}}\right|_{(r_o,c_o)}\right]\tilde{\ddot{c}} + \left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\tilde{\dot{c}} + \left[\left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}\right]\tilde{c} = -\left[\left.\frac{\partial f}{\partial \ddot{r}}\right|_{(r_o,c_o)}\right]\tilde{\ddot{r}} - \left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\tilde{\dot{r}} - \left[\left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}\right]\tilde{r}$$

Linearization of a Magnetic Bearing model

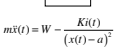
- The magnetic bearing







- The magnetic bearing model



Parameter	Value
Supported Mass Weight, W	1.0 N
Magnet Strength, K	1.0 N.mm2/amp
Gap Constant, a	1.0 mm
Operating Position, xo	2.0 mm

$$m\ddot{x}(t) = W - F(t)$$
$$F(t) = \frac{Ki(t)}{(x-a)^2}$$
(watch units…)

$$m\ddot{x}(t) = W - \frac{Ki(t)}{(x(t)-a)^2}$$
$$\ddot{x}(t) = 1 - \frac{(10^{-6})i(t)}{(x(t)-10^{-3})^2} \quad (N)$$

Linearization of a Magnetic Bearing model

The magnetic bearing model

$$\ddot{x}(t) = 1 - \frac{(10^{-6})i(t)}{(x(t)-10^{-3})^2}$$



1) Identify model input(s) and output(s)

2) Express the model in the $f(\ldots)=0$ form

$$f(\ddot{x},x,i) = \ddot{x}(t) + \frac{(10^{-6})i(t)}{(x(t)-10^{-3})^2} - 1 = 0$$

3) Find an equilibrium for x_o = 2 mm

$$\frac{(10^{-6})i_0}{(x_o-10^{-3})^2} - 1 = 0 \qquad \frac{(10^{-6})i(t)}{(10^{-3})^2} - 1 = 0 \qquad i_o = 1 \text{ (A)}$$

Parameter	Value
Supported Mass Weight, W	1.0 N
Magnet Strength, K	1.0 N.mm2/amp
Gap Constant, a	1.0 mm
Operating Position, xo	2.0 mm

Linearization of a Magnetic Bearing model

4) Taylor Expansion about (i_o, x_o)=(1A, 2 mm)

$$f(\ddot{x},x,i) = \ddot{x}(t) + \frac{(10^{-6})i(t)}{(x(t)-10^{-3})^2} - 1 = 0$$

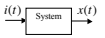
$$f(\ddot{x},x,i) = 0 \cong f(\ddot{x}_o,x_o,i_o) + \frac{\partial f}{\partial \ddot{x}}\bigg|_{(\ddot{x}_o,i_o)}(\ddot{x}-\ddot{x}_o) + \frac{\partial f}{\partial x}\bigg|_{(\ddot{x}_o,i_o)}(x-x_o) + \frac{\partial f}{\partial i}\bigg|_{(\ddot{x}_o,i_o)}(i-i_o)$$



$$0 = 0 + (1)(\ddot{x}-\ddot{x}_o) + \left(\frac{-2(10^{-6})i_o}{(x_o-10^{-3})^3}\bigg|_{(\ddot{x}_o,i_o)}\right)(x-x_o) + \left(\frac{(10^{-6})}{(x_o-10^{-3})^2}\bigg|_{(\ddot{x}_o,i_o)}\right)(i-i_o)$$

$$0 = 0 + (1)(\ddot{x}-\ddot{x}_o) + \left(\frac{-2(10^{-6})(1)}{(10^{-3})^3}\right)(x-x_o) + \left(\frac{(10^{-6})}{(10^{-3})^2}\right)(i-i_o)$$

$$0 = (\ddot{x}-\ddot{x}_o) - (2x10^3)(x-x_o) + (1)(i-i_o)$$



5) Change Variables

$$\tilde{\ddot{x}} = (\ddot{x}-\ddot{x}_o), \quad \tilde{x} = (x-x_o), \quad \tilde{i} = (i-i_o)$$

Parameter	Value
Supported Mass Weight, W	1.0 N
Magnet Strength, K	1.0 N.mm2/amp
Gap Constant, a	1.0 mm
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Linearization of a Magnetic Bearing model

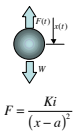
- Rewrite the Taylor series becomes

$$\left.\begin{aligned} 0 &= (\ddot{x}-\ddot{x}_o) - (2x10^3)(x-x_o) + (1)(i-i_o) \\ \tilde{\ddot{x}} &= (\ddot{x}-\ddot{x}_o), \quad \tilde{x} = (x-x_o), \quad \tilde{i} = (i-i_o) \end{aligned}\right\} \Rightarrow \quad 0 = \tilde{\ddot{x}} - (2x10^3)\tilde{x} + \tilde{i}$$

- And the linearized model is

$$\tilde{\ddot{x}} - (2x10^3)\tilde{x} = -\tilde{i}$$





$$F = \frac{Ki}{(x-a)^2}$$

System Responses

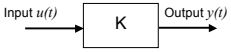
Chapter 4

- Responses of Linear Time-invariant Systems
  - Important input functions
    - Step Response:
      - Constant input, "easy to make", clear transient
    - Ramp Response:
      - Constant change in input, process control
    - Sinusoidal Response:
      - Constant Frequency, for vibration
      - Enough frequencies -> complete system description
    - All have observed properties that engineers use

0th Order systems

- Not in the book, but …
- Much of Engineering design
  - Done with "instantaneous" algebraic models
- The model is a simple "Gain"
- Output is always the input times a constant
  - The model is easy, so is the response computation

$$\frac{Y(s)}{U(s)} = K \quad \text{where } K \text{ is a constant}$$



1st Order Time Response

- 1st order systems have a single derivative of the output variable
- The TF has a first order polynomial in the denominator

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0}{s+a_0} \quad \text{OR} \quad G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$$

- For Engineering Conversation…
  - the second is more common - has physical meaning

1st Order Time Response
(Model has two parameters)

- For Engineering Conversation…
  - the second is more common - has physical meaning

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s+a_0} \quad \text{OR} \quad G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$$

  - Time Constant, τ indicates "speed of response"
  - (Steady-State) Gain K indicates "strength of response"

$$\tau = 1/a_0 \quad \text{and} \quad K = b_0/a_0$$

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1st Order Step Response

- For the 1st order system with TF

$$\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$$

- And step unit input

$$R(s)=\frac{1}{s}$$

- The output is…

$$C(s)=\left(\frac{K}{\tau s+1}\right)\frac{1}{s}=\frac{K}{s(\tau s+1)}=\frac{K/\tau}{s(s+1/\tau)}=\frac{K}{s}+\frac{-K}{(s+1/\tau)}$$

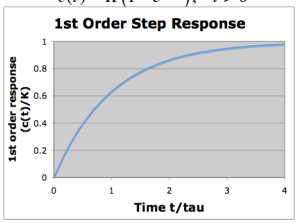
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1st Order Step Response

- The output is…

$$C(s)=\left(\frac{K}{\tau s+1}\right)\frac{1}{s}=\frac{K}{s(\tau s+1)}=\frac{K/\tau}{s(s+1/\tau)}=\frac{K}{s}+\frac{-K}{(s+1/\tau)}$$

$$c(t)=K\left(1-e^{-t/\tau}\right), \quad t>0$$



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Summary and Exercises

- Modeling of
  - Nonlinear systems with linearization
- Next
  - 2nd Order System Time Response

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