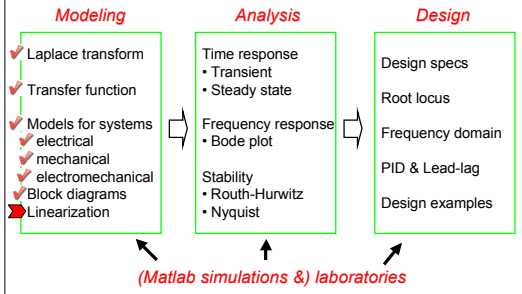
ME451: Control Systems

Lecture 10
Linearization

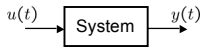
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Course roadmap



What is a linear system?

- A system having ***Principle of Superposition***



$$\left.\begin{array}{l} u_1(t) \rightarrow y_1(t) \\ u_2(t) \rightarrow y_2(t) \end{array}\right\} \Rightarrow \alpha_1 u_1(t) + \alpha_2 u_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t) \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

A nonlinear system does not satisfy the principle of superposition.

Linear systems

- Easier to understand and obtain solutions
- Linear ordinary differential equations (ODEs),
  - Homogeneous solution and particular solution
  - Transient solution and steady state solution
  - Solution caused by initial values, and forced solution
- Add many simple solutions to get more complex ones (use superposition!)
- Easy to check the Stability of stationary states (Laplace Transform)

Which is Linear

- Which of these equations is Linear
  - They must obey superposition, test them

$$y = 5x$$

$$y = 5x + 3$$

$$y = 5x^2 + 3x$$

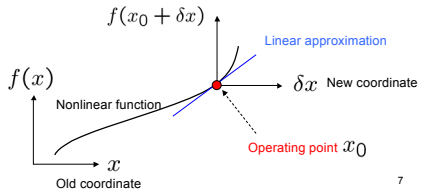
$$y = 5\cos(x)$$

Why linearization?

- Real systems are inherently nonlinear. (Linear systems do not exist!) *Ex.* f(t)=Kx(t), v(t)=Ri(t)
- TF models are only for linear time-invariant (LTI) systems.
- Many control analysis/design techniques are available for linear systems.
- Nonlinear systems are difficult to deal with mathematically.
- Often we linearize nonlinear systems before analysis and design. How?

How to linearize it?

- Nonlinearity can be approximated by a linear function for small deviations δx around an operating point x_0
- Use a Taylor series expansion



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Linearization

- Nonlinear system: $\dot{x} = f(x,u)$
- Let u_0 be a nominal input and let the resultant state be x_0
- Perturbation: $u(\cdot) = u_o(\cdot) + \delta u(\cdot)$
- Resultant perturb: $x(\cdot) = x_o(\cdot) + \delta x(\cdot)$
- Taylor series expansion:

$$f(x,u) = f(x_0,u_0) + \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u + \underbrace{\text{H.O.T}}_{\approx 0}$$

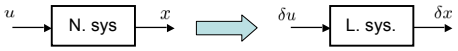
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Linearization (cont.)

$$\dot{x}_0 + \delta\dot{x} = f(x_0,u_0) + \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u$$

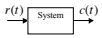
notice that $\dot{x}_0 = f(x_0,u_0)$; hence

$$\delta\dot{x} = \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u$$



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Six Steps to Linearization

1. Identify the system model's inputs and outputs
2. Express the model in the form
   $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) = 0$
   

3. Find an equilibrium where all input and output derivatives are zero and the original model is satisfied at the operating point (r_o, c_o)

   $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) = f(r_o,0,0,\ldots,c_o,0,0,\ldots) = 0$

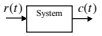
   That is …. Solve $f(r_o,c_o)=0$.
   Typically you are given the output c_o and must find the input r_o

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Six Steps to Linearization

4. Perform a Taylor Series expansion about the operating point retaining only 1st derivative terms.

$$f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) \cong f(r_o,0,0,\ldots,c_o,0,0,\ldots) + \frac{\partial f}{\partial r}\Big|_{(r_o,c_o)}(r-r_o) + \frac{\partial f}{\partial \dot{r}}\Big|_{(r_o,c_o)}(\dot{r}-\dot{r}_o) + \frac{\partial f}{\partial \ddot{r}}\Big|_{(r_o,c_o)}(\ddot{r}-\ddot{r}_o) + \ldots + \frac{\partial f}{\partial c}\Big|_{(r_o,c_o)}(c-c_o) + \frac{\partial f}{\partial \dot{c}}\Big|_{(r_o,c_o)}(\dot{c}-\dot{c}_o) + \frac{\partial f}{\partial \ddot{r}}\Big|_{(r_o,c_o)}(\ddot{c}-\ddot{c}_o) + \ldots$$



5. Change variables

$$\tilde{r} = (r-r_o), \dot{\tilde{r}} = (\dot{r}-\dot{r}_o), \ddot{\tilde{r}} = (\ddot{r}-\ddot{r}_o), etc.$$
$$\tilde{c} = (c-c_o), \dot{\tilde{c}} = (\dot{c}-\dot{c}_o), \ddot{\tilde{c}} = (\ddot{c}-\ddot{c}_o), etc.$$

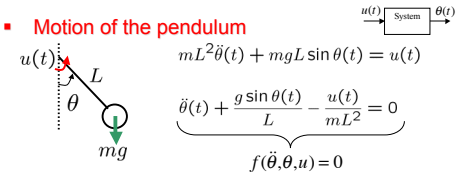
6. Rewrite the Taylor Expansion in the new variables

$$\left[\frac{\partial f}{\partial \ddot{r}}\Big|_{(r_o,c_o)}\right]\ddot{\tilde{c}} + \left[\frac{\partial f}{\partial \dot{r}}\Big|_{(r_o,c_o)}\right]\dot{\tilde{c}} + \left[\frac{\partial f}{\partial c}\Big|_{(r_o,c_o)}\right]\tilde{c} = -\left[\frac{\partial f}{\partial \ddot{r}}\Big|_{(r_o,c_o)}\right]\ddot{\tilde{r}} - \left[\frac{\partial f}{\partial \dot{r}}\Big|_{(r_o,c_o)}\right]\dot{\tilde{r}} - \left[\frac{\partial f}{\partial r}\Big|_{(r_o,c_o)}\right]\tilde{r}$$

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Linearization of a pendulum model

- Motion of the pendulum



- Linearize it at $\theta_0 = \pi$ (Defines output at O.P.)
- Find u_0 $\quad \ddot{\pi} + \frac{g\sin\pi}{L} - \frac{u_0}{mL^2} = 0 \rightarrow u_0 = 0$

$$\theta = \theta_0 + \delta\theta = \pi + \delta\theta$$
$$u = u_0 + \delta u = 0 + \delta u$$

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Linearization of a pendulum model (cont')

- Taylor series expansion $f(\ddot{\theta},\theta,u)=0$ at $(\theta_o,u_o)=(\pi,0)$

$\left.\frac{\partial f}{\partial \ddot{\theta}}\right|_{\substack{\theta=\pi\\u=0}}=1$ $\left.\frac{\partial f(\theta,u)}{\partial \theta}\right|_{\substack{\theta=\pi\\u=0}}=\left.\frac{g\cos\theta}{L}\right|_{\theta=\pi}=-\frac{g}{L}$

$\left.\frac{\partial f(\theta,u)}{\partial u}\right|_{\substack{\theta=\pi\\u=0}}=-\frac{1}{mL^2}$

$$f(\ddot{\theta},\theta,u)=0\cong\left.\frac{\partial f}{\partial \ddot{\theta}}\right|_{\substack{\theta=\pi\\u=0}}(\ddot{\theta}-\ddot{\theta}_o)+\left.\frac{\partial f}{\partial \theta}\right|_{\substack{\theta=\pi\\u=0}}(\theta-\theta_o)+\left.\frac{\partial f}{\partial u}\right|_{\substack{\theta=\pi\\u=0}}(u-u_o)$$

- Change Variables

$\delta\ddot{\theta}=(\ddot{\theta}-\ddot{\theta}_o)$ $\delta\theta=(\theta-\theta_o)$ $\delta u=(u-u_o)$

$$\delta\ddot{\theta}-\frac{g}{L}\delta\theta-\frac{1}{mL^2}\delta u=0$$

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Summary and Exercises

- Modeling of
  - Nonlinear systems with linearization
- Next
  - More linearization examples

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