Controller Principle

Suppose the vehicle is given an orientation change and a constraint that the vehicle will have a constant velocity. The closed loop schematic is shown in Fig. 1:

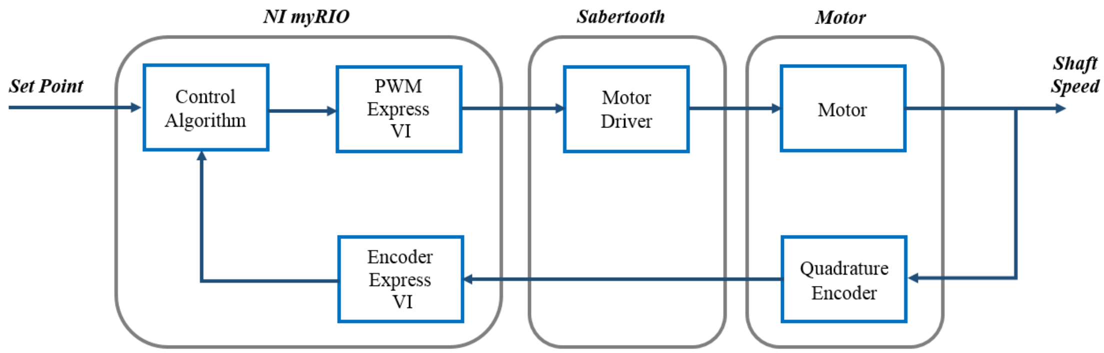

Fig. 1 Internal Controller Block Diagram of the Vehicle

The Set Point is the desired shaft speed of the motor, and the Control Algorithm is commonly a PID controller. The Quadrature Encoder serves as a sensor and the output of the Encoder Express VI can be counts to measure the output shaft angle of or count rate to measure the output shaft speed.

The vehicle is composed of two DC motors, one on each wheel, for which the following block diagram of Fig. 2 can be used to obtain a relation between their inputs and outputs:

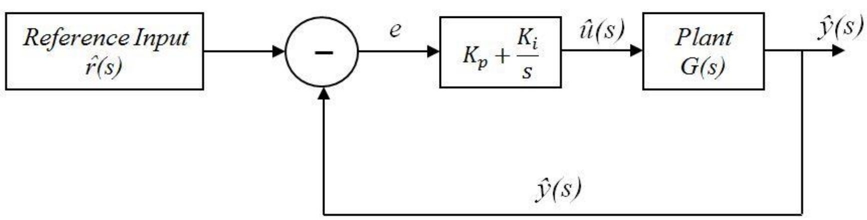

Fig. 2 Internal Controller Block Diagram of the Vehicle

where the plant represents the DC motors with the following transfer function:

$$G(s) = \frac{K}{\tau s + 1} \tag{1}$$

In addition the PI controller has the form:

$$\hat{u}(s) = \left(K_p + \frac{K_i}{s}\right)(\hat{r}(s) - \hat{y}(s)) \tag{2}$$

in which K_p and K_i are the proportional and integral gains, respectively, and $\hat{u}(s)$, $\hat{r}(s)$, and $\hat{y}(s)$ are the control input, reference input, and the DC motor output, respectively. Using the above definitions and rearranging the equations, the closed-loop transfer function, $\hat{y}(s) = H(s).\hat{r}(s)$ can be find as:

$$\begin{aligned}
&\hat{y}(s) = G(s)\hat{u}(s) \\
&\rightarrow \hat{y}(s) = G(s)\left[\left(K_p + \frac{K_i}{s}\right)(\hat{r}(s) - \hat{y}(s))\right] \\
&\hat{y}(s) = G(s)\left(K_p + \frac{K_i}{s}\right)\hat{r}(s) - G(s)\left(K_p + \frac{K_i}{s}\right)\hat{y}(s) \\
&\left[1 + G(s)\left(K_p + \frac{K_i}{s}\right)\right]\hat{y}(s) = G(s)\left(K_p + \frac{K_i}{s}\right)\hat{r}(s) \\
&\hat{y}(s) = \frac{G(s)\left(K_p + \frac{K_i}{s}\right)}{1 + G(s)\left(K_p + \frac{K_i}{s}\right)}\hat{r}(s)
\end{aligned} \tag{3}$$

Substituting G(s) from Eq. (3) and doing some simplifications, H(s) can obtained as:

$$\begin{aligned}
&\hat{y}(s) = \frac{\frac{K}{(\tau s + 1)}\left(K_p + \frac{K_i}{s}\right)}{1 + \frac{K}{(\tau s + 1)}\left(K_p + \frac{K_i}{s}\right)}\hat{r}(s) \times \frac{s(\tau s + 1)}{s(\tau s + 1)} \\
&\hat{y}(s) = \left[\frac{K(K_p s + K_i)}{s(\tau s + 1) + K(K_p s + K_i)}\right]\hat{r}(s) \\
&\Rightarrow H(s) = \frac{K(K_p s + K_i)}{\tau s^2 + (KK_p + 1)s + KK_i}
\end{aligned} \tag{4}$$

The objective of this Lab is to choose K_p and K_i so that control system has fast control with satisfactory stability.

In the following part we will discuss and derive K_p and K_i values that will hopefully meet the design objectives. Note that the steady-state error can be found using the final value theorem. Moreover, if $H(0)=1$, then there is no steady-state error. Since the numerator has a zero, $H(s)=\frac{KK_p s + KK_i}{....}$, the FVT holds.

Instead of analyzing the transfer function, let's use a heuristic approach called the Ziegler-Nickols (ZN) this can be used if TF is not known. In 1942 Ziegler and Nichols described simple mathematical procedures for tuning PID controllers. These procedures are now accepted as standard in control systems practice. The ZN formulae for specifying the controllers are based on plant step responses.

Steps to determine PID controller parameters:

1. Reduce the integrator and derivative gains to 0.
2. Increase K_p from 0 to some critical value at which there is a transitioning from a stable response to an unstable one.
3. Note the value K_{pu} and the corresponding period of marginally stable oscillation, T_u.
4. The controller gains are now specified as follows:

$$K_p \approx 0.4K_{pu}\ ,\ K_i \approx \frac{K_p}{0.8T_u} = 0.5\frac{K_p}{T_u} \tag{5}$$

Note that if the stability of the control loop is poor, try to improve the stability by decreasing K_p, for example a 20% decrease. For the current configuration of the two wheel vehicle the approximate values of $K_{pu} \approx 5.5$ *and* $T_u \approx 0.025\ s$ can be obtained.

Discrete Time PID Controller Design

In this section we assume that a continuous-time controller is given as a transfer function, *C(s)* is described as follows:

$$C(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s} \tag{6}$$

It is desired to find an algorithm (difference equation) for microprocessors so that the digital controller *C(z)* approximates the continuous-time controller. To look for methods to approximate the transfer function, we start by examining a simpler question: what is the equivalent of the differential operator (d/d*t* or *s*) in terms of the shift operator (*z*)?

The discrete time integral and derivatives can be defined as below:

$$y(k) \approx \frac{f(k) - f(k-1)}{T_s} \tag{7}$$

$$\int_0^{k*T_s} y(t)\,\mathrm{d}t = y(k) \approx y(k-1) + \frac{f(k) + f(k-1)}{2} * T_s \tag{8}$$

where T_s is the sampling time. Z -transform, like the Laplace transform, is an indispensable mathematical tool for the design, analysis and monitoring of systems. The z-transform is the discrete-time counter-part of the Laplace transform and a generalization of the Fourier transform of a sampled signal. Like Laplace transform the z-transform allows insight into the transient behavior, the steady state behavior, and the stability of discrete-time systems. The z-transform can be defined as:

$$X[z] = Z\{x[k]\} = \sum_{k=0}^{\infty} x[k] z^{-k} \tag{9}$$

$$z = Ae^{j\theta} = A(\cos\theta + j\sin\theta) \tag{10}$$

And the following time shifting property exists in the z-domain:

$$Z\{x[k-n]\} = z^{-n} X(z) \tag{11}$$

where

$$Z\{x[k]\} = X[z] \tag{12}$$

Using the above definitions the discrete transfer function of derivatives can be presented as:

$$y(k) = \frac{f(k) - f(k-1)}{T_s} \tag{13}$$

$$Y[z] = \frac{F[z] - z^{-1}F[z]}{T_s} \tag{14}$$

$$\frac{Y[z]}{F[z]} = \frac{z-1}{z\,T_s} \tag{15}$$

And the discrete transfer function of integrals can be described as:

$$y(k) = y(k-1) + \frac{f(k) + f(k-1)}{2} T_s \tag{16}$$

$$Y[z]\left(1 - z^{-1}\right) = \frac{T_s}{2} F[z]\left(1 + z^{-1}\right) \tag{17}$$

$$\frac{Y[z]}{F[z]} = \frac{T_s}{2} \frac{z+1}{z-1} \tag{18}$$

Using the above definitions for discrete transfer function of integrals and derivatives, the discrete-time PID controller transfer function can be obtained as:

$$C(z) = \frac{U[z]}{E[z]} = K_p + K_i \frac{T_s}{2} \frac{z+1}{z-1} + K_d \frac{z-1}{z\,T_s} \tag{19}$$

$$\frac{U[z]}{E[z]} = \frac{K_p(z^2 - z) + K_i \frac{T_s}{2}(z^2 + z) + \frac{K_d}{T_s}(z^2 - 2z + 1)}{z^2 - z} \tag{20}$$

$$\frac{U[z]}{E[z]} = \frac{\left(K_p + K_i \frac{T_s}{2} + \frac{K_d}{T_s}\right) z^2 + \left(-K_p + K_i \frac{T_s}{2} - \frac{2K_d}{T_s}\right) z + \frac{K_d}{T_s}}{z^2 - z} \tag{21}$$

Implementation of a digital PID yields to:

$$\frac{U[z]}{E[z]} = \frac{\left(K_p + K_i \frac{T_s}{2} + \frac{K_d}{T_s}\right) + \left(-K_p + K_i \frac{T_s}{2} - \frac{2K_d}{T_s}\right) z^{-1} + \frac{K_d}{T_s} z^{-2}}{1 - z^{-1}} \tag{22}$$

$$U[z] = z^{-1}U[z] + a\,E[z] + b\,z^{-1}E[z] + c\,z^{-2}E[z] \tag{23}$$

$$u[k] = u[k-1] + a\,e[k] + b\,e[k-1] + c\,e[k-2] \tag{24}$$

where

$$a = \left(K_p + K_i \frac{T_s}{2} + \frac{K_d}{T_s} \right)$$

$$b = \left(-K_p + K_i \frac{T_s}{2} - \frac{2K_d}{T_s} \right) \quad (25)$$

$$c = \frac{K_d}{T_s}$$

Finally, the control signal saturation should also be checked. Is control signal within limits? If, not limit the control signal by the maximum value that can be achieved.

$$\begin{gathered} \text{if } \ fabs(u) \geq u_{\max} \\ u = u / fabs(u) * u_{\max} \end{gathered} \quad (26)$$