ME 451
Mechanical Vibrations
Laboratory Manual

A. G. Haddow
haddow@egr.msu.edu
Edited by G. D. Recktenwald

Last updated, Fall 2015

Contents

1 Free Vibration **1**

1.1 Theory 2

1.1.1 Free Vibration, Undamped 2

1.1.2 Free Vibration, Damped 3

1.1.3 The Mass-Spring-Dashpot (MSD) Model 4

1.2 Logarithmic Decrement 7

1.3 Force Transmitted to Base 9

1.4 Phase Plane 9

1.5 Laboratory Procedure 9

2 System Parameter Identification **12**

2.1 Parameters 13

2.2 MSD Model Parameters 13

2.2.1 Masses and spring constants 13

2.2.2 Damping coefficent 14

2.3 Measurement Errors 15

2.4 Differential Error Analysis 16

2.4.1 Example: Finding the stiffness of a spring 17

2.5 Laboratory Procedure 19

3 Forced Vibration **20**

3.1 Direct Harmonic Forcing 21

3.2 Base Excitation 23

3.3 Rotating Unbalance 25

3.4 Laboratory Procedure 28

4 Modal Analysis **29**

4.1 Introduction 30

4.2 Mode Shapes and Principal Coordinates 31

4.3 Three Degree of Freedom System 34

4.4 Continuous Systems 36

4.5 Laboratory Procedure 37

Laboratory 1

Free Vibration

Summary

This laboratory introduces the basic principles involved in free vibration. The apparatus consists of a spring-mass-damper system that includes three different springs, variable mass, and a variable damper. The laboratory is designed to provide the students with insight into the influence of the parameters involved in the governing equations of the system.

Various experiments will be run during the laboratory period. The students will be expected to calculate data based on the theory presented herein and compare that data with experimental results.

1.1 Theory

1.1.1 Free Vibration, Undamped

Consider a body of mass m supported by a spring of stiffness k, which has negligible inertia (Figure 1.1). Let the mass m be given a downward displacement from the static equilibrium position and released. At some time t the mass will be at a distance x from the equilibrium position and the spring force kx acting on the body will tend to restore it to its equilibrium position. By summing the forces in the vertical direction and assuming

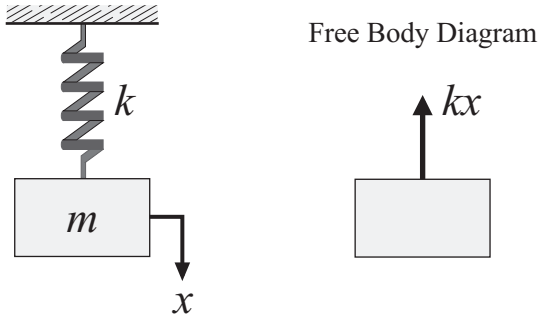


Figure 1.1 Spring-Mass System

motion about the static equilibrium position, $\Sigma F = ma$ yields

$$-kx = m\frac{d^2x}{dt^2} \tag{1.1}$$

or, rearranging

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0 \tag{1.2}$$

where

$$\omega_n^2 = \frac{k}{m}.$$

If k and m are in standard units; the natural frequency of the system ω_n will have units of rad/s.

The solution to the above equation is of the form

$$x(t) = A\cos\omega_n t + B\sin\omega_n t, \tag{1.3}$$

where A and B are constants of integration which are determined from the initial conditions, i.e., $x(0)$ and $\dot{x}(0)$.

The solution is periodic in period T, where

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{1.4}$$

The frequency f, measured in Hz, is then given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_s}} \tag{1.5}$$

1.1.2 Free Vibration, Damped

Consider a body of mass m supported by a spring of stiffness k and attached to a dash pot whose resistance may be considered proportional to the relative velocity (Figure 1.2). Let the mass m be given a downward (i.e., positive) displacement from the equilibrium position and released. At some time t the mass will be at a distance x from the equilibrium position. The spring force kx acting on the body will tend to restore it to its equilibrium position and the damper force tending to oppose motion will be $c\frac{dx}{dt}$ where c is the viscous damping coefficient.

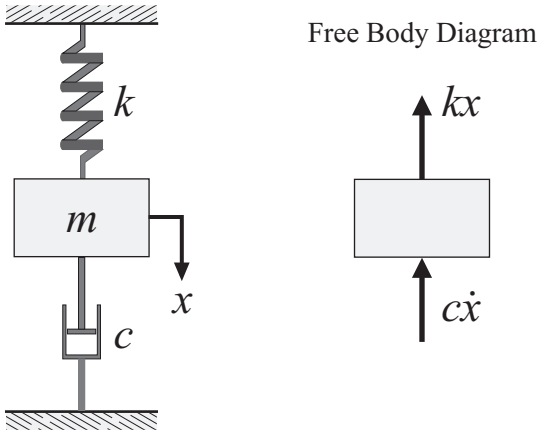


Figure 1.2 Spring-Mass-Damper System

By summing the forces in the vertical direction and assuming motion about the static equilibrium position (refer to the free body diagram), the equation of motion is

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx \tag{1.6}$$

This equation can be rewritten as follows:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = 0 \tag{1.7}$$

with the following definitions:

$$\omega_n^2 = \frac{k}{m} \quad \text{and} \quad 2\zeta\omega_n = \frac{c}{m}$$

where ζ is the damping ratio for the given spring-mass-damper system. For this case

$$\zeta = \frac{c}{2m\omega_n} \tag{1.8}$$

The form of the solution of this differential equation depends on the value of ζ. There are three cases:

Case I	$\zeta > 1$	the system is *overdamped*
Case II	$\zeta = 1$	the system is *critically damped*
Case III	$\zeta < 1$	the system is *underdamped*

In cases I & II no oscillation occurs. We are usually more interested in Case III which does support oscillatory behavior. It can be shown that the solution in this case is given by

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi), \tag{1.9}$$

where A and ϕ are constants of integration found by considering the system's initial conditions and ω_d is the damped natural frequency. It is defined as

$$\omega_d = \omega_n\sqrt{1-\zeta^2}.$$

The general form of this solution is shown in Figure 1.3 along with the solutions associated with Cases I & II and the Case where $\zeta = 0$, the undamped case.

1.1.3 The Mass-Spring-Dashpot (MSD) Model

The system shown in Figure 1.2 is a fundamental and simple model known mass-spring-dashpot (MSD) model. Contrary to the perception of many

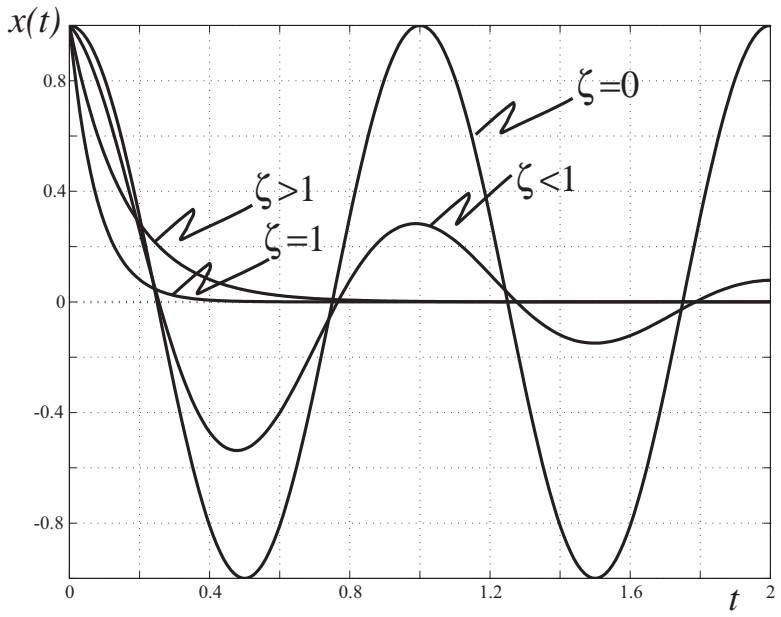


Figure 1.3 Free Vibration Response for Various Damping Ratios

young engineers, this simple model is useful not only for academic purposes, but also as the basic engineering building block for the analysis and understanding of a large class of vibrating systems. Even in the analysis of complex physical systems, the total behavior can be thought of as a linear combination of mass-spring-dashpot systems, each system being known as a vibration mode.

Another common misconception is that the model shown in Figure 1.2 is related to only a particular physical system which consists of a lumped mass attached to a tension-compression spring with a dashpot acting in parallel. Indeed, many students have difficulty in appreciating that the systems given in Figure 1.4 can all be represented by the basic mass-spring-dashpot model. Of course, this generalization involves different sets of assumptions and approximations. Therefore, in some of those systems the MSD model may represent only part of the dynamics or the results may have limited accuracy. However, the MSD model is a reasonable representation to begin with; and it is the building block to understanding vibrations of a more complex nature.

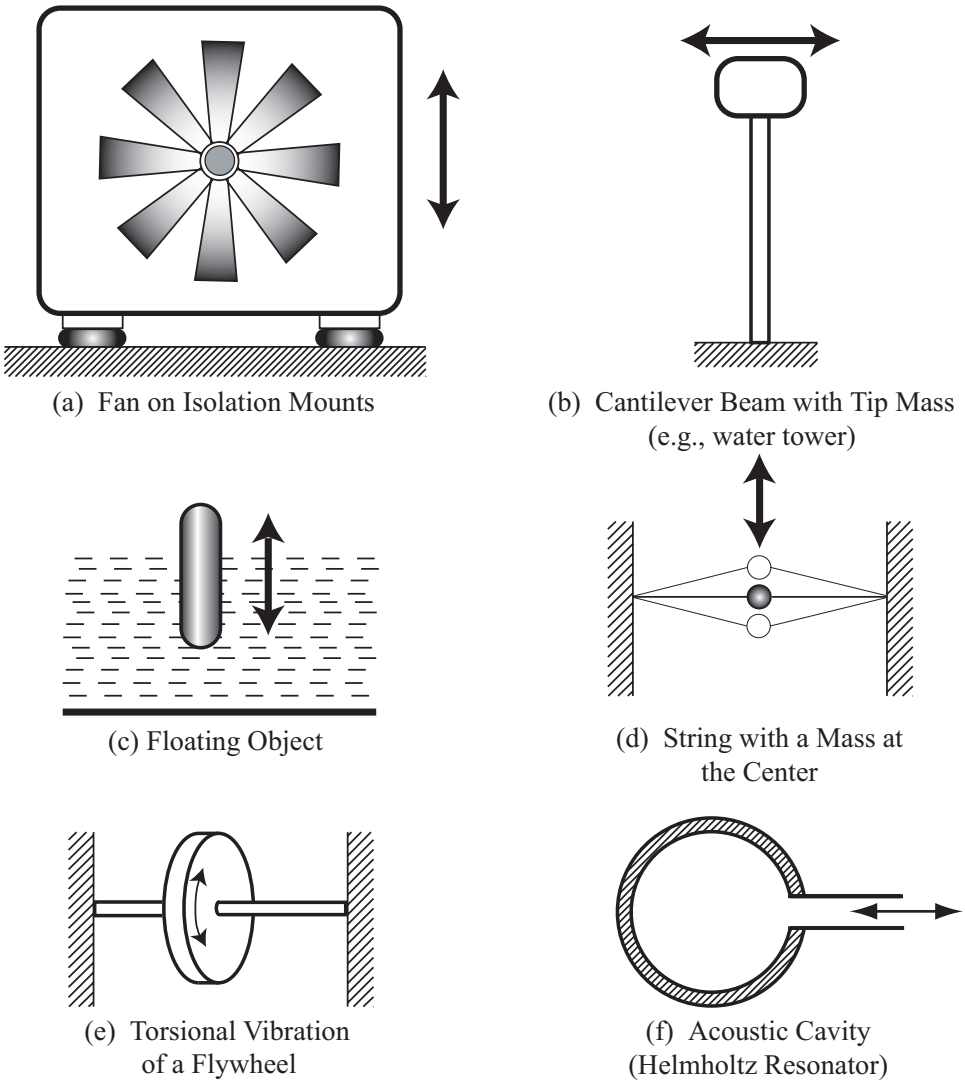


Figure 1.4 Examples of Physical Systems that Could be Modeled as a SDOF System

1.2 Logarithmic Decrement

For the case where $\zeta < 1$ we have a solution of the form as shown in Figure 1.5. The logarithmic decrement δ is defined as the natural logarithm of the ratio of any two successive peaks, i.e., $\delta = \ln\left(\frac{X_1}{X_2}\right)$ (See Figure 1.5.) It is possible to relate δ to the damping ratio ζ and hence, this gives us a convenient method for experimentally obtaining a measure of the damping in a system. The relationship between δ and ζ is obtained as follows. With reference to Figure 1.5, τ_d is defined as $\tau_d \equiv \frac{2\pi}{\omega_d}$, i.e., the period of the function $\sin(\omega_d t + \phi)$. Then, defining X_1 as the amplitude of the oscillation at time $t = t_1$, we have

$$X_1 = Ae^{-\zeta\omega_n t_1}\sin(\omega_d t_1 + \phi). \tag{1.10}$$

It follows that X_{1+s}, the amplitude of the displacement s periods from the first peak, is

$$X_{1+s} = Ae^{-\zeta\omega_n(t_1+s\tau_d)}\sin\left[\omega_d(t_1 + s\tau_d) + \phi\right], \qquad s = 1, 2, 3, \ldots \tag{1.11}$$

Next, consider the ratio of two peaks, s periods apart, i.e.,

$$\frac{X_1}{X_{1+s}} = e^{\zeta\omega_n s\tau_d}. \tag{1.12}$$

Finally, taking the natural logarithm of both sides, yields

$$\ln\left(\frac{X_1}{X_{1+s}}\right) = \zeta\omega_n s\tau_d = \frac{2\pi s\zeta}{\sqrt{1-\zeta^2}}. \tag{1.13}$$

If $s = 1$ we have

$$\delta = \ln\left(\frac{X_1}{X_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}. \tag{1.14}$$

In fact this is true for any two successive peaks, e.g.,

$$\delta = \ln\left(\frac{X_3}{X_4}\right).$$

In practice, the damping ratio ζ is often small and therefore the equation above can be approximated as

$$\delta = 2\pi\zeta. \tag{1.15}$$

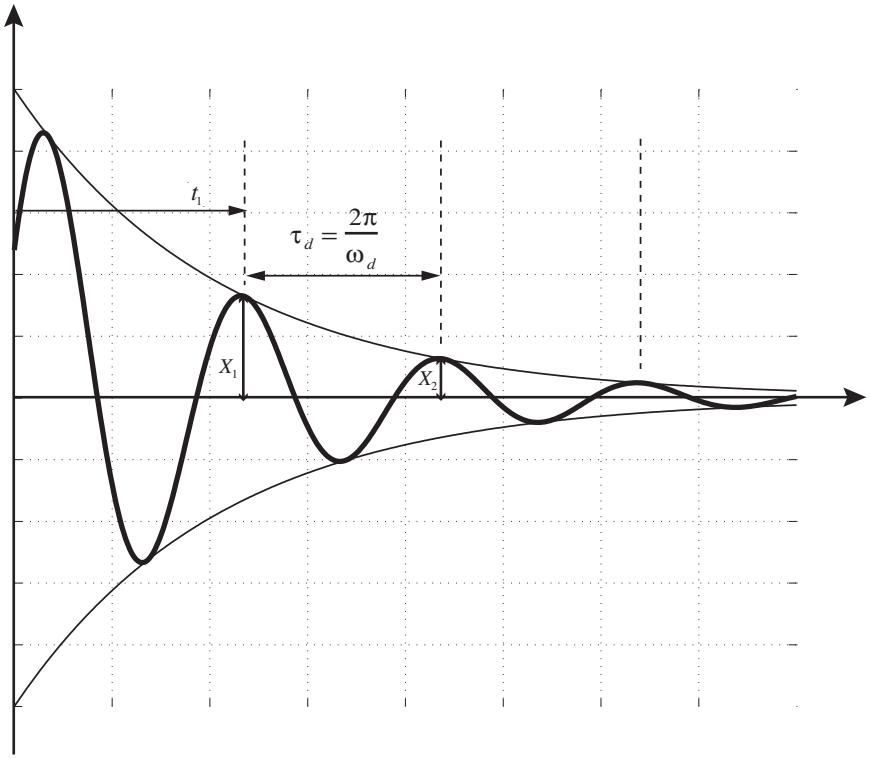


Figure 1.5 Time History of Decaying Oscillation

This approximation gives good results for ζ values up to approximately 0.2.

When trying to evaluate ζ, accuracy improves if we measure more than two peaks of the oscillation (i.e. $s > 1$). Therefore, assuming a small value for ζ, we have

$$\ln\left(\frac{X_1}{X_{1+s}}\right) = 2s\pi\zeta, \tag{1.16}$$

i.e., if we plot $\ln\left(\frac{X_1}{X_{1+s}}\right)$ versus s, then a straight line through the points will have a gradient (or slope) of $2\pi\zeta$. This procedure provides a means of averaging the data and has the effect of minimizing the errors made in reading the amplitudes. It also indicates if viscous damping is or is not a good model for the type of damping actually present. (Why?)

1.3 Force Transmitted to Base

As shown in Figure 1.2, there are two points that are attached to the ground; one is the end of the spring and the other is the end of the damper. (Note, it doesn't matter if the spring is attached above the mass as in Figure 1.2, or below the mass and thus to the side of the damper.) The total force transmitted to the base is the summation of the forces that are in each of these elements. However, care must be taken in the addition process, for the two forces will not be in phase, i.e., when one reaches its maximum value, the other will not be at its maximum. In fact, for harmonic motion, one is 90 degrees out of phase with the other.

1.4 Phase Plane

A graph showing the velocity versus displacement is known as a phase-plane or state-space plot. The time, t, can not be seen directly as it is a parameter that increases as one moves along the curve. Figure 1.6 shows examples of phase-plane plots, where the arrows show the direction of increasing time. There are examples for under-damped, critically-damped, and over-damped free vibration systems and an under-damped system subjected to sinusoidal forcing (this latter phase-plane is included here for completeness and will be more fully discussed in the *Forced Vibration* laboratory). These types of plots are useful in evaluating characteristics of a system, such as its stability, and give an overview of the general motion that ensues. They can also be used to directly observe if a motion is periodic, such as the forced case in Figure 1.6.

1.5 Laboratory Procedure

This laboratory experience uses the three cart apparatus on the lab bench. Two of the carts will be fixed and the middle cart will oscillate freely. We will use the ECP control program to record displacement of the cart. The data can be displayed as a snapshot, in realtime, or exported and displayed in matlab.

The laboratory procedure manual details the step by step process of working through the laboratory. Be sure to fill in the shortform as you procede through the experiments.

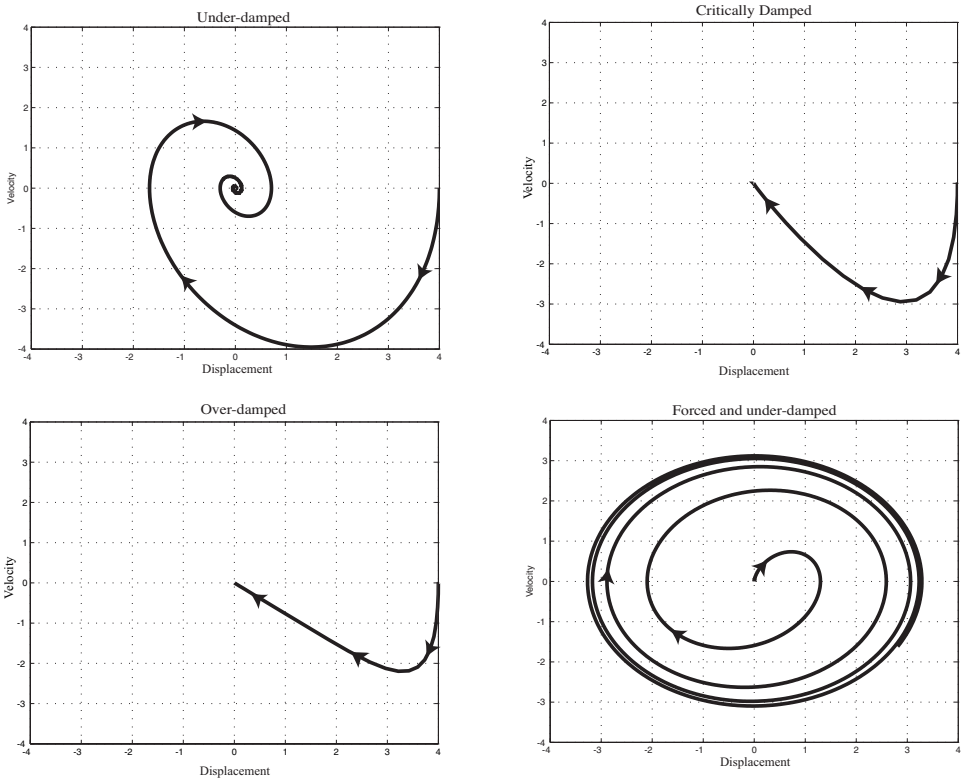


Figure 1.6 Samples of Phase Plane Plots

The parameters for this system are as follows. The spring rates are:

$$\begin{aligned} k_{weak} &= 175N/m \\ k_{medium} &= 400N/m \\ k_{stiff} &= 800N/m \end{aligned}$$

Each slotted mass is $500g$, the cart mass is $700g$.

<u>CAUTION</u>: Be sure to read the lab manual safety tips.

The system has sliding friction, but we will assume it is negligable for the purposes of this lab. A dashpot can be attached to the system to introduce viscous damping. The amount of damping can then be altered by turning

the thumb screw up or down, thus decreasing or increasing how easily the air can exit or enter the piston.

There are three components to this laboratory experience. The first deals with studying how the displacement-time graphs of the system change when one varies the stiffness, mass, and damping values. You will first be asked to look at the undamped case for three different combinations of stiffness and mass. At least two different springs should be tested and use a different mass for each case. Next you shall introduce damping and collect data that will enable you to quantify just how much damping is present, and what type of damping it is.

In Part II of the laboratory experience, you will investigate force-time graphs and consider how they are added in order to obtain the total force transmitted to the base. Finally, in Part III, phase-plane graphs will be obtained.

You are also encouraged to try additional combinations of system parameters and initial conditions in order to gain a physical appreciation for the behavior of a linear, second order, differential equation.

Laboratory 2

System Parameter Identification

Summary

This laboratory introduces some of the basic principles involved in charactrizing a system by experimentation. In the previous lab, the model and system parameters were provided in the laboratory manual. In practice, parameters are found as part of the experimental process. Models are validated by the same process.

In this laboratory the student will perform a series of experiments designed to charactize the system they are studying. They will find the mass, spring constants, and damping coefficients. The student is expected to determine measurement errors and be able to calculate how those errors affect the derived parameters.

The parameter values found in this lab will be used in future labs.

2.1 Parameters

What is a parameter? A parameter is a measurable feature that partially defines a system. Parameters can be simple, like the mass of an object, or more complex, like the Reynolds number for a fluid stream. In both cases the parameter is well defined and has a known affect on the system. Parameters can be found by direct measurement, indirect measurement, or statistical estimation.

Direct measurement involves finding a parameter value using a method that is specific to that value type. An example of direct measurement is finding the mass of an object by using a scale.

A measurement is indirect when the parameter value is found by measuring a related value. An example of an indirect measurement is using a strain gage to measure stress. Stress cannot be measured directly so strain, which is related to stress, by Hooke's Law is measured instead.

Parameter Estimation is a type of indirect measurement. It involves using statistics and optimization to match a model and parameters to the behavior of a system. This method is particularly helpful when device used to measure the system is susceptible to random noise or error. Parameter Estimation is beyond the scope of this laboratory.

2.2 MSD Model Parameters

The Mass-Spring-Damper model discussed in Lab 1, Eq 1.6, has three measurable parameters; m, the mass of the cart and weights, k, the spring stiffness, and c, the damping coefficient.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{1.6}$$

These parameters could be combinations of other parameters. For example the mass, m, is a combination of the mass of the cart, m_c, the mass of the weights m_w, and the mass of the spring m_s.

2.2.1 Masses and spring constants

The mass of the MSD system can be measured directly. The spring stiffness is typically measured indirectly by measuring the displacement due to a known force and using Hookes law $F = kx$.

The methods described above for measuring m and k require the appratus to be disassembled. This is not always possible or even desirable. Knowledge of the theory of vibrating systems provides a means of measuring m and k in the assembled state. To do this one needs to find the natural frequency and use the fact that $\omega_n^2 = k/m$.

Since this is only one equation and we have two unknowns, at least two experiments must be performed. Since the mass of the weights, m_w is known, the weights can be added (or removed) to find several linearly independent equations, for example:

$$\text{Cart with no masses :} \quad \omega_n^2 = \frac{k}{m_c} \tag{2.1}$$

$$\text{Cart with 4 masses :} \quad \omega_n^2 = \frac{k}{4m_w + m_c} \tag{2.2}$$

One might expect these two methods of measuring $m$and k to yield the same results. However, the first method ignores the mass of the spring. The second measurement combines the spring mass with the mass of the cart. Measuring the system parameters in an assembled state often provides a better representation of what the system 'sees'.

2.2.2 Damping coefficent

In Lab 1 the damping ratio was computed by the log-decrement method, Eq. 2.3.

$$\zeta \approx \frac{1}{2\pi n} \ln\left(\frac{X_0}{X_n}\right) \tag{2.3}$$

where ζ was defined by

$$\zeta = \frac{c}{2m\omega_n} \equiv \frac{c}{2\sqrt{km}}. \tag{2.4}$$

The approximation for ζ is valid if the damping coefficent is relatively small, $\zeta < 0.1$. The damping coefficent is obtained by two measurements of distance and requires the previous measurement of mass and stiffness.

2.3 Measurement Errors

A vital aspect of any experiment involves understanding the errors involved. Measurement errors take three forms: Systematic, Random, and Resolution. Errors can affect both the precision and the accuracy of a measurement. Accuracy is a measure of how close the measurement is to the correct answer. Precision is how consistent the measurements are to each other, Fig. 2.1.

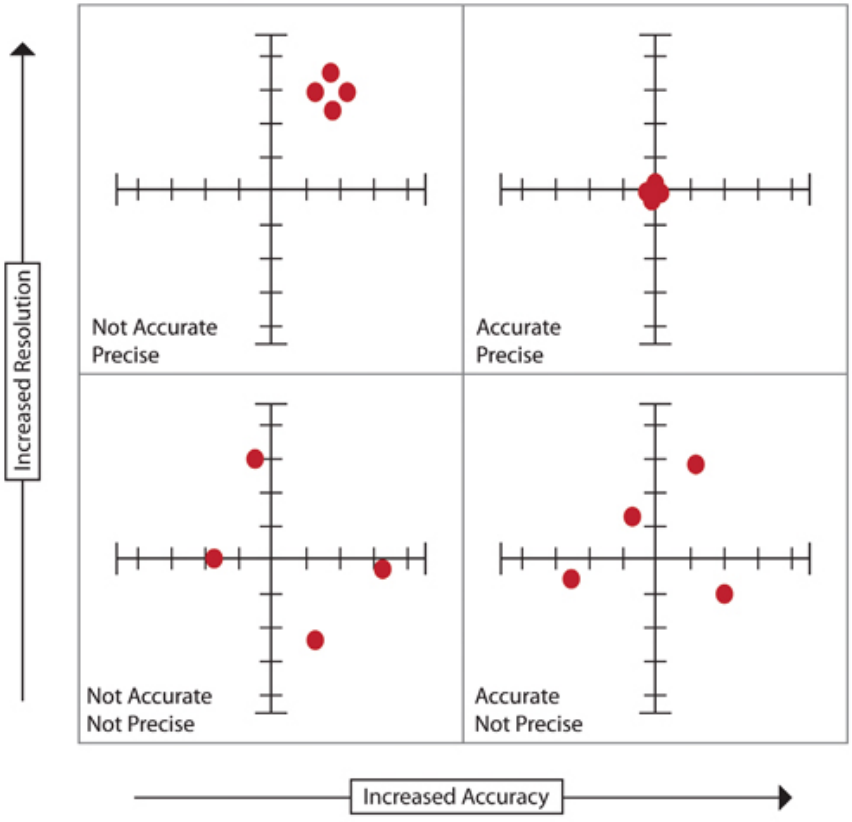


Figure 2.1 Accuracy vs. Precision. (source: vimin.org)

Systematic errors are the result of inaccuracy that is inherent in the system, measurement device, or person doing the measurement. These errors can appear in the data as an offset (axis shift), scaling factor (slope error), or nonlinear adjustment. Unfortunately these errors affect the accuracy of your measurement are difficult to detect.

Random errors are caused by unknown changes in the experiment like noise in an electronic measuring device. If the results are Gaussian, then the measurement error falls off $\propto \frac{1}{\sqrt{n}}$, where n is the number of samples taken. Since random errors only affect the precision of the measurement they can be detected and accounted for by repeating the same experiment multiple times or taking multiple samples.

Resolution errors are the result of limitations in the measuring method. A meter stick marked in centimeters cannot provide millimeter level precision. For analog measuring devices, it is best to assume that a measurement is precise to within half the resolution, so in the case of the meter stick, $\Delta x = \pm 0.5cm$.

Industial measurement devices usually come with data sheets that provide the user with an estimate of the accuracy it can provide. Knowing the accuracy of a measurement is just as important as the measurement itself.

2.4 Differential Error Analysis

The previous section discussed measurement errors. Knowledge of measurement errors is sufficient for direct measurements. However, this laboratory exercise focuses on indirect measurements. To understand the error in the system parameters one has to understand how the errors in measurements propogate through an equation. This topic is called "Differential Error Analysis".

Recall from calculus that a function of one independent variable $y = f(x)$ can be expressed as a Taylor series.

$$y = y_0 + \frac{df(x)}{dx}\Delta x + ... \tag{2.5}$$

To find the change in y, Δy, the value y_0 is moved to the left hand side of the equation.

$$\Delta y = y - y_0 = \frac{df(x)}{dx}\Delta x + ... \tag{2.6}$$

For a function of two (or more) independent variables $z = f(x, y)$, Δz is found by computing the multidimensional Taylor series, and moving z_0 to the left hand side of the equation.

$$\Delta z = z - z_0 = \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y + ... \tag{2.7}$$

To get the absolute value of Δz both sides are squared and the square root is taken.

$$|\Delta z| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial f}{\partial y}\right)^2 (\Delta y)^2 + 2\left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right)\Delta x \Delta y} \tag{2.8}$$

In Equation 2.8, Δx and Δy are the absolute error of the experimental measurement (which are assumed to be symmetric and small in this treatment). Additionally, the cross term, with $\Delta y \Delta x$ is set to zero because we assume that there is no correlation between the measurement errors.

This result is generalized below to include any number of parameters.

Differential Error Analysis:

Given a set of experimental data and small uncorrelated errors of the form:

$$x_1 \pm \Delta x_1,\ x_2 \pm \Delta x_2,\ \ldots\ ,\ x_n \pm \Delta x_n$$

The error in derived quantity y, where $y = f(x_1, x_2, ..., x_n)$ can be approximated as:

$$\Delta y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 (\Delta x_1)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 (\Delta x_2)^2 + ... + \left(\frac{\partial f}{\partial x_n}\right)^2 (\Delta x_n)^2} \tag{2.9}$$

2.4.1 Example: Finding the stiffness of a spring

A student seeking to measure the stiffness of a spring designs an experiment that measures the deflection of the spring, x, caused by an applied force, F. The student is using a measuring device with centimeter markings and a force transducer accurate to 1 Newton. The measured force is $40N$ and the measured deflection is 10cm.

The data is recorded in the lab notebook with appropriate notation of the measurement errors, units, and significant figures.

$$\text{deflection}: \quad x = 10.0 \pm 0.5cm \tag{2.10}$$

$$\text{force}: \quad F = 40 \pm 1N \tag{2.11}$$

The student then computes the stiffness, k, paying careful attention to significant digits, Eq 2.12.

$$k = \frac{F}{x} = 4.0\frac{N}{cm} \tag{2.12}$$

To compute the error, Δk, the student calculates the partial derivatives of the function $f(x, F) = F/x$ and plugs the result into Equation 2.9.

$$\begin{aligned}
\Delta k &= \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial f}{\partial F}\right)^2 (\Delta F)^2} \\
&= \sqrt{\left(\frac{-F}{x^2}\right)^2 (\Delta x)^2 + \left(\frac{1}{x}\right)^2 (\Delta F)^2} \\
&= \sqrt{\left(\frac{-40}{10^2}\right)^2 (0.5)^2 + \left(\frac{1}{10}\right)^2 (1)^2} \\
&= 0.22\ \frac{N}{cm}
\end{aligned} \tag{2.13}$$

The student marks in their lab book that the stiffness of the spring is

$$k = 4.0 \pm 0.2\ \frac{N}{cm} \tag{2.14}$$

After determining that the error in k is too large, the student goes back to equation 2.13 and notes the following:

1. Reducing deflection error to $\Delta x = 0.25$ makes $\Delta k = 0.14$.

2. Reducing force error to $\Delta F = 0.5$ makes $\Delta k = 0.21$.

The student concludes that obtaining a meter stick with half-centimeter markings is the most efficient way to obtain a more precise value of the stiffness, k.

2.5 Laboratory Procedure

This laboratory contains a set of experiments similar to Lab 1. However, the focus on this laboratory is to use the experiments (and theory) to characterize the system. Care should be taken to minimize the errors in each measurement and students should make several measurements to verify that the random error is small. Students will need to calculate the error for derived quantities. This last step can be done outside of the lab.

After completing the lab, students will need to make a copy of the last page of their lab report as a reference for Labs 3 and 4.

Laboratory 3

Forced Vibration

Summary

This laboratory demonstrates the behavior of a sinusoidally forced, single degree-of-freedom, spring-mass-damper system. Three different types of force can be imposed upon the system: one arising from direct forcing, one from base excitation, and another from a rotating unbalance. These systems behave in very similar ways. Many of the physical parameters of the system can be changed, and the influence of these changes on the system's response can then be studied.

This lab will focus on studying base excitation. Particular attention is given to the phenomenon of resonance, the influence of damping, and the phase relationship between input and output terms.

3.1 Direct Harmonic Forcing

In the *Free Vibration* laboratory, we focused on the mass-spring-dashpot model and emphasized its importance due to its wide range of applications. In the following, we will continue working on the MSD model but now our focus will be on the *forced* vibration response of the system. Consider the

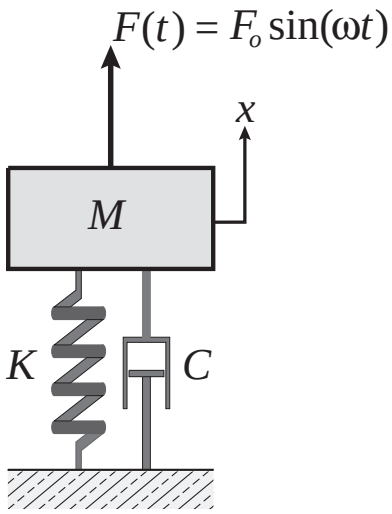


Figure 3.1 A Mass-Spring-Dashpot Model with Direct Harmonic Forcing

MSD model shown in Figure 3.1. Based on a free body diagram and Newton's second law, the motion of the mass about its static equilibrium position is governed by the following non-homogenous, second-order, ordinary differential equation

$$M\ddot{x} + C\dot{x} + Kx = F_o \sin \omega t, \tag{3.1}$$

where M, C, and K are, respectively, the effective mass, damping, and stiffness of the system. It is important to note the use of the word *effective*. Depending on the actual physical system being modeled, these parameters may contain more than a simple mass, damping and stiffness term. However, if the ODE is of the same form as Equation (3.1), then the solution can easily be found by comparison.

It is sometimes useful to re-write Equation (3.1) by dividing throughout by

M to obtain

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_o}{M}\sin\omega t, \tag{3.2}$$

where the terms ζ, and ω_n are defined as usual. Great care should be taken when considering the magnitude of the *forcing* term on the right hand side of the equation, F_o/M. Often this is re-defined as a single parameter, but it must be realized that it no longer has the units of *Force* (it actually has units of acceleration). If we define

$$f_o = \frac{F_o}{M}$$

then the amplitude, X, and the phase, ϕ, associated with the steady-state solution,
$x_{ss}(t) = X\sin(\omega t - \phi)$, can be written in a number of different ways

$$X = \frac{F_o/K}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{f_o/\omega_n^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{F_o}{\sqrt{(K - M\omega^2)^2 + (C\omega)^2}} \tag{3.3}$$

and

$$\tan\phi = \frac{2\zeta r}{(1-r^2)} = \frac{C\omega}{(K - M\omega^2)} \tag{3.4}$$

where r is defined in the usual way. A plot of the amplitude, X, (or its nondimensional form) versus the forcing frequency (or its nondimensional form) is known as a $Frequency Response Function$ (FRF). An example of such a function is shown in Figure 3.2

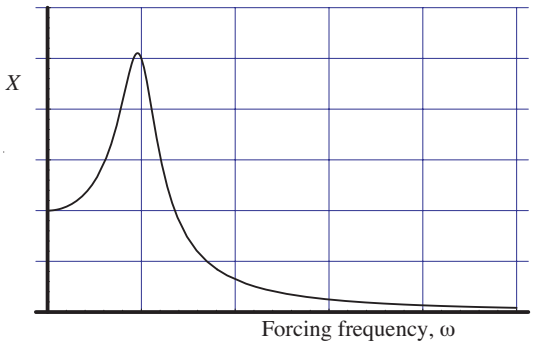


Figure 3.2 A Typical Frequency Response Function for a Single Degree-of-Freedom System

Finally, it should be noted that the complete solution of the ODE comprises two parts: the transient solution (obtained by solving the homogeneous equation) and the steady state solution. The transient solution $x_{tr}(t)$ is known to be

$$x_{tr}(t) = X_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \phi_1) \qquad \text{iff} \quad \zeta < 1 \tag{3.5}$$

where X_1 and ϕ_1 are evaluated from the system's initial conditions and ω_d, the damped natural frequency, is

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{3.6}$$

The completed solution is then

$$x(t) = x_{tr}(t) + x_{ss}(t). \tag{3.7}$$

3.2 Base Excitation

Consider the system shown in Figure 3.3. The equation of motion governing the absolute displacement of the mass is given by

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}, \tag{3.8}$$

where m is the mass of the system, c is the viscous damping coefficient, and k is the linear spring stiffness. If we define a new coordinate z such that $z = x - y$, i.e. it is the displacement of the mass relative to the base, then Equation (3.8) can be rewritten as

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}. \tag{3.9}$$

If the base motion is given as $y(t) = Y\sin\omega t$ then Equation (3.9) becomes

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = Y\omega^2\sin\omega t \tag{3.10}$$

where

$$\zeta = \frac{c}{c_c}$$

is the damping ratio,

$$c_c = 2\sqrt{km}$$

is the critical damping coefficient, and

$$\omega_n = \sqrt{\frac{k}{m}}$$

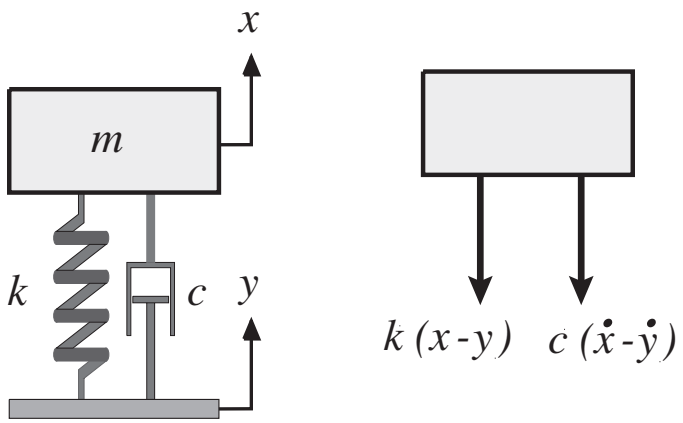


Figure 3.3 SDOF Mass-Spring-Dashpot Model with Base Excitation

is the natural frequency.

The complete solution of this equation comprises two parts: the transient solution (obtained by solving the homogeneous equation) and the steady state solution. The transient solution z_{tr} can be shown to be

$$z_{tr}(t) = X_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \phi_1) \qquad \text{iff} \quad \zeta < 1$$

where X_1 and ϕ_1 are evaluated from the system's initial conditions and ω_d, the damped natural frequency, is

$$\omega_d = \omega_n\sqrt{1-\zeta^2}.$$

It must be emphasized that the constants X_1 and ϕ_1 are very different from the amplitude Z and phase ϕ. The latter are *fixed* by the system's parameters (see below), whereas the former are found from the initial conditions of the problem.

The steady state solution z_{ss} can be shown to be

$$z_{ss}(t) = Z\sin(\omega t - \phi) \tag{3.11}$$

where

$$Z = \frac{Yr^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3.12}$$

and

$$\tan\phi = \frac{2\zeta r}{(1-r^2)} \tag{3.13}$$

with

$$r = \frac{\omega}{\omega_n}.$$

The complete solution is then

$$z(t) = z_{tr}(t) + z_{ss}(t). \tag{3.14}$$

However, if the damping is nonzero, the transient solution will decay to zero as time increases and we will simply be left with the steady state solution

$$z(t) = Z\sin(\omega t - \phi). \tag{3.15}$$

To obtain the *absolute* motion of the mass, $x(t)$, we can either solve Equation (3.8) directly or use the relationship that $x(t) = z(t) + y(t)$ to obtain

$$x(t) = X\sin(\omega t - \psi) \tag{3.16}$$

where

$$X = Y\sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \tag{3.17}$$

and

$$\tan\psi = \frac{2r^3\zeta}{(1 - r^2) + (2\zeta r)^2}. \tag{3.18}$$

The variation of the steady-state amplitudes Z and X can be plotted as a function of the forcing frequency. It is convenient to non-dimensionalize such plots so that they become independent of a system's parameters. This is accomplished by plotting Equation (3.11) as Z/Y versus r in Figure 3.4(a) and Equation (3.17) as X/Y versus r in Figure 3.4(b), where $r = \frac{\omega}{\omega_n}$, with ω_n constant. The functions are evaluated for various values of damping ratios, ζ. The associated phase angles ϕ and ψ are also shown in these figures.

3.3 Rotating Unbalance

Unbalance in rotating machines is a very common source of vibration excitation. Consider the system shown in Figure 3.5. Mass m denotes the unbalance that is rotating at an angular velocity of ω and at a radius of e. The total mass of the system is M. (Note that the m and M notation is different than used in the previous section. However, it is consistent with

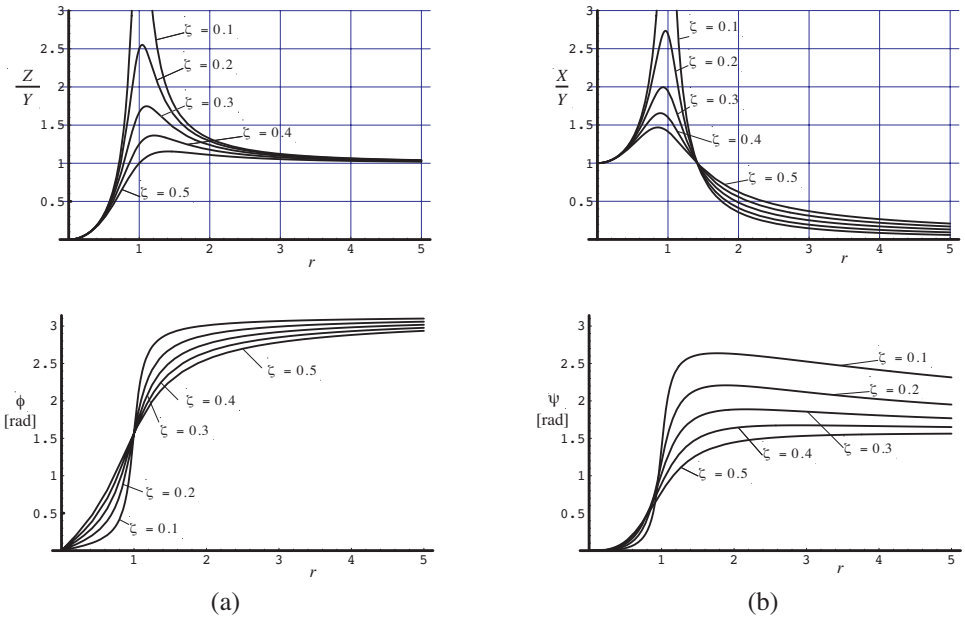


Figure 3.4 Frequency Response Functions for Base Excitation. (a) Z/Y and the associated phase, ϕ. (b) X/Y and the associated phase, ψ.

the class notes and the text book.) The equation of motion of this system can be shown to be

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t. \tag{3.19}$$

The right hand side of the equation originates from the angular acceleration of the rotating unbalance in the x direction. As before, it is more convenient to re-write Equation (3.19) as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega^2 x = \frac{me\omega^2}{M} \sin \omega t \tag{3.20}$$

where

$$\zeta = \frac{c}{c_c}$$

is the damping ratio,

$$c_c = 2\sqrt{kM}$$

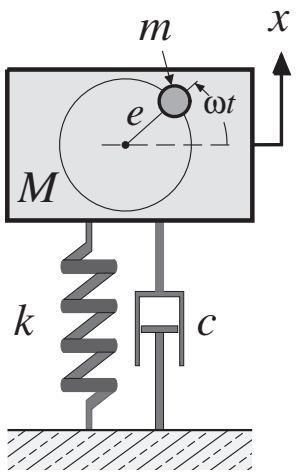


Figure 3.5 SDOF Rotating Unbalance System

is the critical damping coefficient, and

$$\omega_n = \sqrt{\frac{k}{M}}$$

is the natural frequency.

We note that at a fixed frequency, the right side of equation Equation (3.19) can be rewritten in same form as refeq:EOMstandard, the the standard harmonically forced MSD model of Section 3.1.

Comparing Equation (3.20) to Equation (3.9), we see that they are of the same form and so the steady state component of the solution can immediately be written as

$$x(t) = X\sin(\omega t - \phi), \tag{3.21}$$

where

$$X = \frac{mer^2}{M\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3.22}$$

and where the phase angle is unchanged and is given by Equation (3.13)

$$\tan\phi = \frac{2\zeta r}{(1-r^2)}.$$

Thus, the frequency response function for this system is the same as that depicted in Figure 3.4(a), except that Z is replaced by X and Y is replaced by me/M.

3.4 Laboratory Procedure

This laboratory experience uses the three cart apparatus on the lab bench. Carts 1 and 2 will be free to move. Cart 1 is connected to a motor that can provide force or displacement excitation. We will use the ECP control program to provide the forcing and record displacement of the carts. The data can be displayed as a snapshot, in realtime, or exported and displayed in matlab.

The laboratory procedure manual details the step by step process of working through the laboratory. Be sure to fill in the shortform as you procede through the experiments.

The parameters for this system were found in lab 2.

CAUTION:

1. **Be sure to read the lab manual safety tips.**
2. **Bring a copy of your Lab 2 - Appendix with system parameter data.**
3. **Be sure to use the same piece of equipment you used last time.**

There are three components to this laboratory experience. The first deals with studying how the system amplitude changes as a function of the driving frequency. You will be asked to adjust the input frequency between $r = 0.6$ and 2.0 and analyse the response. Next you will study how the system phase changes as a function of r. Finally, you will use the industry standard sine-sweep (Chirp) input and fast fourier transform (FFT) analysis. You will compare the results obtained by both methods.

You are also encouraged to try additional combinations of system parameters and initial conditions in order to gain a physical appreciation for the behavior of a linear, second order, differential equation.

Laboratory 4

Modal Analysis

Summary

This laboratory demonstrates a number of concepts that generalize to multi-degree-of-freedom systems. A three degree of freedom system is used to physically show what mode shapes are and to demonstrate that more than one natural frequency exists. Methods for experimentally obtaining mode shapes and natural frequencies are explored and the concept of principal (or modal) coordinates is introduced.

4.1 Introduction

Although some real systems can be modeled accurately as single-degree-of-freedom systems, many structures exist where a higher number of degrees-of-freedom must be used. To fully understand the dynamical behavior of, for example, an aircraft, thousands of degrees of freedom have to be employed. It is possible to analyze this even before the structure is made (e.g. by using Finite Element Methods), but usually a model of the structure will still be built to ensure that the behavior is as expected. In addition, vibration analysis often has to be carried out on an existing structure to better understand why it is vibrating and how best to minimize the vibration.

From a vibration standpoint, two of the most useful quantities that can be obtained experimentally are the system's natural frequencies and the associated mode shapes. The mode shape is the shape that the system oscillates in when it is responding at, or close to, a natural frequency. There are a variety of standard tests that can be used to obtain this (and other) information. We shall be using a form of the **Frequency Sweep** technique where the test structure is excited sinusoidally and the frequency of the excitation is slowly varied between predetermined limits. The response of the system at various points on the structure is then recorded. Other common techniques used to excite the system are shock loading (often an instrumented hammer is used) or a random noise source.

In the frequency sweep method the forcing frequency is slowly varied and the system will pass through various natural frequencies. At these frequencies the system responds in a resonant (large) manner. Hence, if a continuous recording is taken of some coordinate, say x_i, then the Frequency Response Function for the *amplitude* of this coordinate, X_i will have a form as shown in Figure 4.1, where the ω_i 's are the system's natural frequencies. If the experiment were now to be repeated and data collected from a different point, say x_j, then another response function could be obtained. It would have peaks at the same values of ω but the height of the peaks would, in general, be different. From this information the shape that the system vibrates in can be deduced. In practice, complex computer programs are used to analyze the different frequency response functions and to calculate and present the various mode shapes and their associated natural frequencies. Damping can also be determined.

In this laboratory experience, we will carry out a simple form of a frequency sweep on a three degree-of-freedom model. The forcing frequency will be

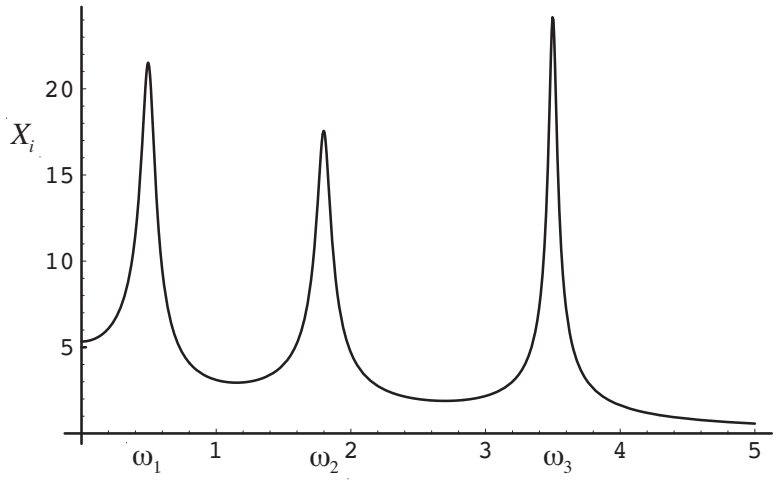


Figure 4.1 A Typical *Frequency Response Function* for a Three Degree-of-Freedom System. The Steady-state Amplitude of the i^{th} Coordinate versus the Forcing Frequency.

manually swept and stopped at, or at least close to, the two different natural frequencies. The steady-state responses will then be measured at these frequencies. Therefore, instead of recording the steady-state amplitude at *all* the forcing frequencies and thus obtain a function similar to Figure 4.1, we will simply record specific points on the frequency response functions. This will still allow us to obtain the mode shapes and estimate the natural frequencies.

To better obtain the natural frequencies, we will also apply simple impact testing on the system and then perform an FFT analysis of the measured response.

4.2 Mode Shapes and Principal Coordinates

In the experimental set-up used in the laboratory (see Figure 4.2), the mode shapes can easily be viewed by the human eye. It is even possible to obtain a good quantitative estimate of how much one coordinate moves relative

to the other. For a more complex structure and/or one that is not vibrating with such a large amplitude, it may be impossible to view the mode shapes directly. One then has to rely on *measurements* taken at a number of different locations on the system. For example, in our system an accurate measurement of the mode shape can be obtained by measuring the amplitudes of the $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$ signals while the forcing frequency is held

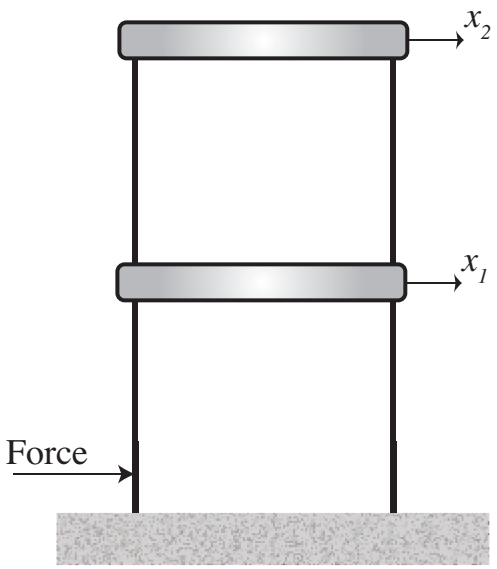


Figure 4.2 A Schematic of a 2 DOF System.

constant first at one of the natural frequencies and then at the other. (Note, will the ratio of $\ddot{x}_1(t) : \ddot{x}_2(t)$ be the same as $x_1(t) : x_2(t)$?) If the frequency response functions had already been obtained, they could have been used in place of taking measurements at any one particular frequency. Such a procedure can be generalized for more than two degrees of freedom.

The concept of principal coordinates (sometimes called modal coordinates) is useful while studying multi-degree-of-freedom systems. These coordinates measure how much of each mode is present at any given time. It can be shown that they are a linear combination of the generalized coordinates $x_i(t)$ (see the class text book or class notes). Another way of stating this is that the coordinates, p_1 and p_2 are related to the generalized coordinates, x_1 and

x_2 by the following matrix equation:

$$\vec{x} = [\mathbf{P}]\vec{p} \tag{4.1}$$

where $[\mathbf{P}]$ is known as the modal matrix and performs the appropriate linear transformation from the generalized coordinates, $\vec{x}$, to the principal coordinates, $\vec{p}$. It can be further shown that each *column* of the matrix $[\mathbf{P}]$ is in fact a mode shape of the system. This holds true even for N degree-of-freedom system, in which case the matrix $[\mathbf{P}]$ would be of dimension $N \times N$.

The use of principal coordinates also allow the equations of motion to be uncoupled. This results in a vast mathematical simplification. Instead of having to simultaneously solve N coupled differential equations, we only have to solve N *uncoupled* single degree-of-freedom equations. Note that these equations can all be solved independently of each other, hence the mathematical simplification.

By way of example, consider a 2 degree-of-freedom system (see Figure 4.2). The equation of motion using the generalized coordinates x_i, can be shown to be of the form:

$$m_1\ddot{x}_1 + k_{11}x_1 + k_{12}x_2 = F_1 \sin\omega t \tag{4.2}$$
$$m_2\ddot{x}_2 + k_{21}x_1 + k_{22}x_2 = F_2 \sin\omega t \tag{4.3}$$

where we have assumed the damping to be negligible. In order to analytically find $x_1(t)$ and $x_2(t)$, these equations have to be solved simultaneously. For more than a two-degree-of-freedom system, this becomes extremely hard to do. However, if we use the coordinate transformation given by Equation (4.1) to transform Equation (4.2) and Equation (4.3) into the principal coordinates, the resulting equations will be uncoupled. The equation for p_1 will not contain the variable p_2 and the equation for p_2 will not contain variable p_1, i.e.,

$$\ddot{p}_1 + \omega_1^2 p_1 = f_1(t) \tag{4.4}$$
$$\ddot{p}_2 + \omega_2^2 p_2 = f_2(t) \tag{4.5}$$

where $\vec{f}(t) = [\mathbf{P}]^T\vec{F}(t)$ and ω_1 and ω_2 are the natural frequencies of the system. (Strictly, to obtain this specific form of the uncoupled equations, the modal matrix $[\mathbf{P}]$ has to be correctly normalized.) Each principal coordinate thus acts like a single-degree-of-freedom system, whose solution can be found easily. Note that this same approach generalizes for N DOF systems.

In practice, all of the aforementioned manipulations can be carried out by commercially available software packages that can efficiently analyze experimental data and automatically output animations of the mode shapes, obtain the modal matrix [**P**], and find the associated damping of each mode. There are thriving consulting companies that specialize in this type of work, where experiments are completed on complex structures and detailed, empirical, equations of motion can be obtained.

4.3 Three Degree of Freedom System

In this lab we will study a 3-mode system 4.3.

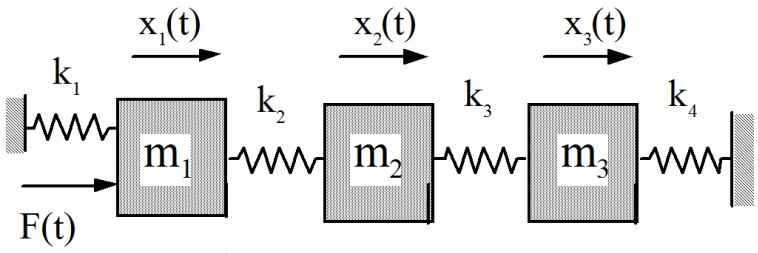


Figure 4.3 A 3-DOF model

In this system the first cart is directly forced and the absolute position of each mass is measured by an encoder.

Using the method of virtual displacement or Free Body Diagrams we can quickly write down the equations of motion for each of the carts. For cart 2 we start with

$$m_1\ddot{x}_2 = -k_2(x_2 - x_1) + k_3(x_3 - x_2) \tag{4.6}$$

which can be collected into vector form

$$m_1\ddot{x}_2 + \begin{bmatrix} -k_2 & (k_2+k_3) & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \tag{4.7}$$

If we repeat the process for carts 1 and 3 and combine the result we obtain a coupled differential equation system in matrix form, Eq. 4.8.

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \\ 0 \end{bmatrix} \tag{4.8}$$

We often simplify this type of equation by defining a mass matrix, $\mathbf{M}$, and stiffness matrix, $\mathbf{K}$ to obtain Eq. 4.9. It should be observed that $\mathbf{M}$ and $\mathbf{K}$ are symmetric.

$$\mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = \vec{F}(t) \tag{4.9}$$

In the lab, we will examine a system where the masses and spring constants are all the same. Thus,

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}. \tag{4.10}$$

We are interested in the modal frequencies of this system. To do this we will solve the homogeneous differential equation by assuming a solution in the form of a mode. The spatial variable $\vec{x}$ and its derivatives become

$$\vec{x} = \vec{A}\cos\omega t \tag{4.11}$$

$$\dot{\vec{x}} = -\vec{A}\omega\sin\omega t \tag{4.12}$$

$$\ddot{\vec{x}} = -\vec{A}\omega^2\cos\omega t. \tag{4.13}$$

Plugging this into the homogenized Equation 4.9 leads to

$$\left(-\mathbf{M}\omega^2 + \mathbf{K}\right)\vec{(}A)\cos\omega t = 0. \tag{4.14}$$

Since $\vec{A} = 0$ and $\cos\omega t = 0$ lead to trivial solutions we must set $\det\left[-\mathbf{M}\omega^2 + \mathbf{K}\right] = 0$;

$$\det\begin{bmatrix} -m\omega^2+2k & -k & 0 \\ -k & -m\omega^2+2k & -k \\ 0 & -k & -m\omega^2+2k \end{bmatrix} = 0. \tag{4.15}$$

and solve the resulting cubic function, Eq. 4.16 for ω, Eq. 4.17.

$$\left(-m\omega^2+2k\right)\left((-m\omega^2+2k)^2 - 2k^2\right) = 0 \tag{4.16}$$

$$\begin{aligned} \omega_1 &= \sqrt{2-\sqrt{2}}\sqrt{\frac{k}{m}} \\ \omega_2 &= \sqrt{2}\sqrt{\frac{k}{m}} \\ \omega_3 &= \sqrt{2+\sqrt{2}}\sqrt{\frac{k}{m}} \end{aligned} \tag{4.17}$$

You can use the results from laboratory 2 to estimate the frequencies you expect to see in the system. However, even without physical values, we can say that

$$\frac{\omega_2}{\omega_1} = \frac{\sqrt{2}}{\sqrt{2-\sqrt{2}}} \approx 1.84 \qquad \text{and} \qquad \frac{\omega_3}{\omega_2} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}} \approx 1.31. \tag{4.18}$$

Each of these "eigenfrequencies" correspond to "eigenmodes" or mode shapes. We can find these mode shapes by finding the values of $\vec{A}_i$ that satisfy Equation 4.14. Since the forcing is only on cart 1, it will excite the different modes with a different amount of force. This will only alter the magnitude of the response, not the shape of the mode.

4.4 Continuous Systems

These results can be extended to continuous systems, including multi-dimensional continuous systems, Fig. 4.4. A continuous system will exhibit natural frequencies and mode shapes associated with each natural frequency. The major difference between continuous and discrete systems is that continuous systems will have an infinite number of modes.

A discussion of analytic and numerical methods for finding these mode-shapes and frequencies is beyond the scope of this laboratory. However, the experimental techniques parallel the methods we are using for discrete systems. The system is excited at a range of frequencies by a frequency sweep or impulse, the response of the system is measured and a fast Fourier Transform is used to find the natural frequencies. Mode shapes are continuous so a strobe light is often used to observe shape.

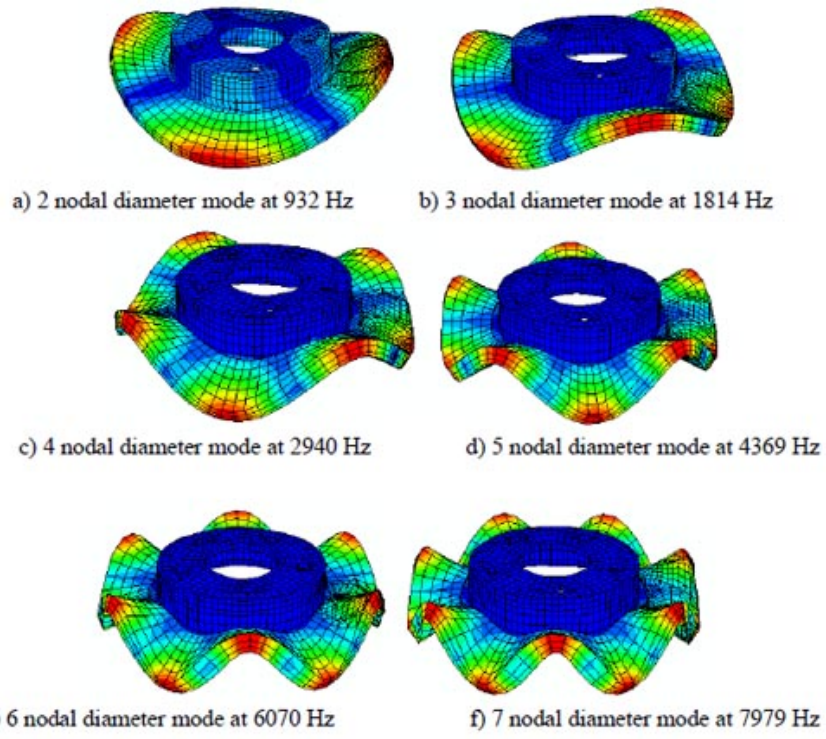


Figure 4.4 Car disc brake: mode shapes and natural frequencies. (ref: Abu-Bakar and Ouyang, "Recent Studies of Car Disc Brake Squeal")

4.5 Laboratory Procedure

This laboratory experience uses the three cart apparatus on the lab bench. You will be using an impulse and sine sweep combined with a FFT code in Matlab to find the modal frequencies of the system. Once you know the frequencies you can dial them in and observe the mode shape of the system in resonance. You will be comparing the values with eachother and the frequencies found in this pre-lab.

Additionally you will experiment with two continuous systems. The first continuous system is a beam with free-free end conditions that is excited at the center. The system will be forced with a frequency sweep. As the frequency changes you are asked to note the natural frequencies and mode

shapes.

The second continuous system is a string. For this system you will force the string manually, trying to find the natural frequencies. Be sure to mark in your lab report who was able to find the highest natural frequency / mode shape.