# ME451: Control Systems

**Lecture 9**

**Stability**

Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University



# Course roadmap

***Modeling***
- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ electromechanical
- ✓ Block diagrams
- ✓ Linearization

***Analysis***
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- ➤ Stability
  - Routh-Hurwitz
  - (Nyquist)

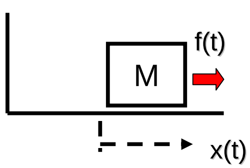
***Design***
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

*(Matlab simulations &) laboratories*

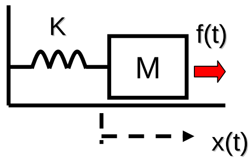


# Simple mechanical examples

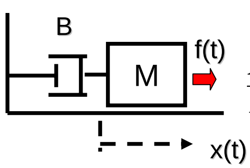
- We want mass to stay at x=0, but wind gave some initial speed (F(t)=0). What will happen?



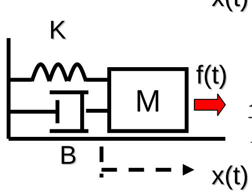

$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$




$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$




$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs}$$




$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs + K}$$

- How to characterize different behaviors with TF?

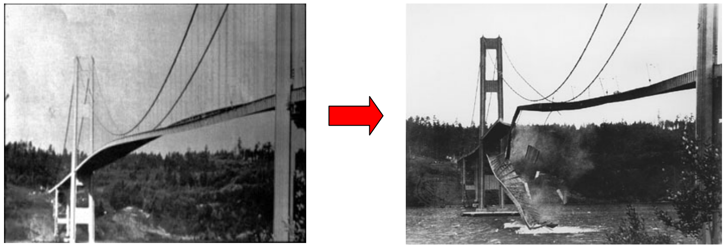


# Stability

- Utmost important specification in control design!
- Unstable systems have to be stabilized by feedback.
- Unstable closed-loop systems are useless.
  - What happens if a system is unstable?
    - may hit mechanical/electrical “stops” (saturation)
    - may break down or burn out

# What happens if a system is unstable?

Tacoma Narrows Bridge (July 1-Nov.7, 1940)



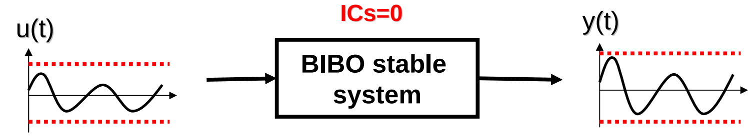
**Wind-induced vibration** **Collapsed!**

2008…

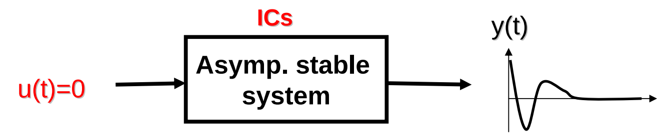


# Mathematical definitions of stability

- **BIBO** (Bounded-Input-Bounded-Output) **stability** :
  *Any bounded input generates a bounded output.*




- **Asymptotic stability** :
  *Any ICs generates y(t) converging to zero.*






# Some terminologies

$$G(s) = \frac{n(s)}{d(s)}$$

**Ex.** $$G(s) = \frac{(s-1)(s+1)}{(s+2)(s^2+1)}$$

- *Zero* : roots of n(s) $\qquad$ $(\text{Zeros of } G) = \pm 1$
- *Pole* : roots of d(s) $\qquad$ $(\text{Poles of } G) = -2, \pm j$
- *Characteristic polynomial* : d(s)
- *Characteristic equation* : d(s)=0



# Stability condition in s-domain

(Proof omitted, and not required)

**For a system represented by a transfer function G(s),**

**system is BIBO stable**

⇕

***All the poles of G(s) are in the open left half of the complex plane.***

⇕

**system is asymptotically stable**

# "Idea" of stability condition

*Example* $y'(t) + \alpha y(t) = u(t), \; y(0) = y_0$

➡ $sY(s) - y(0) + \alpha Y(s) = U(s)$

➡ $Y(s) = \frac{1}{s+\alpha}(U(s) + y(0))$

Asym. Stability: (U(s)=0) $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}y(0)\right\} = e^{-\alpha t}y(0) \to 0 \Leftrightarrow \mathrm{Re}(\alpha) > 0$

BIBO Stability: (y(0)=0) $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\} = \int_0^t g(\tau)u(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}u(t-\tau)d\tau$

$$|y(t)| \le \int_0^t |e^{-\alpha\tau}||u(t-\tau)|d\tau \le \underline{\int_0^t |e^{-\alpha\tau}|d\tau} \cdot u_{max}$$

Bounded if Re(α)>0



# Remarks on stability

- For a general system (nonlinear etc.), BIBO stability condition and asymptotic stability condition are different.
- For linear time-invariant (LTI) systems (to which we can use Laplace transform and we can obtain a transfer function), the conditions happen to be the same.
- In this course, we are interested in only LTI systems, we use simply "stable" to mean both BIBO and asymptotic stability.



# Remarks on stability (cont'd)

- Marginally stable if
  - G(s) has no pole in the open RHP (Right Half Plane), &
  - G(s) has at least one simple pole on $j\omega$-axis, &
  - G(s) has no multiple poles on $j\omega$-axis.

$$G(s) = \frac{1}{s(s^2+4)(s+1)} \qquad G(s) = \frac{1}{s(s^2+4)^2(s+1)}$$

***Marginally stable*** ***NOT marginally stable***

- Unstable if a system is neither stable nor marginally stable.



# Examples

- Repeated poles

$$\mathcal{L}^{-1}\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right] = t\sin\omega t \qquad \mathcal{L}^{-1}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right] = t\cos\omega t$$

$$\cdots = t^2\sin\omega t \qquad \cdots = t^2\cos\omega t$$

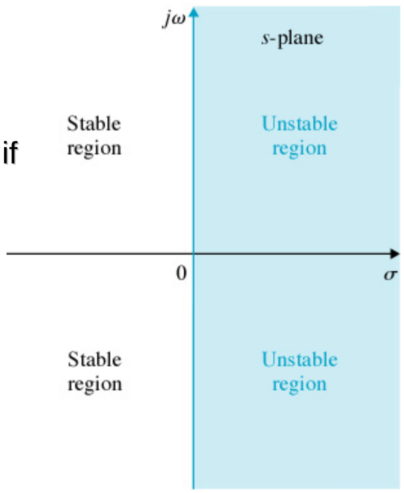
- Does marginal stability imply BIBO stability?
  - TF: $G(s) = \frac{2s}{(s^2+1)}$
  - Pick $u(t) = \sin t \;\; \overrightarrow{\mathcal{L}} \;\; U(s) = \frac{1}{(s^2+1)}$
  - Output $\mathcal{L}^{-1}\left[Y(s) = G(s)U(s) = \frac{2s}{(s^2+1)^2}\right] = t\sin t$

# Stability summary

Let $s_i$ be poles of G.
Then, G is …

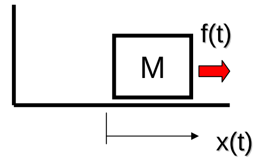
- (BIBO, asymptotically) stable if $Re(s_i)<0$ for all i.
- marginally stable if
  - $Re(s_i)<=0$ for all i, and
  - simple root for $Re(s_i)=0$
- unstable if

  it is neither stable nor marginally stable.



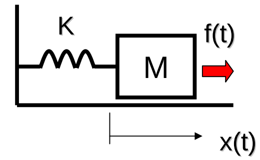



# Mechanical examples: revisited



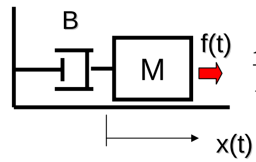

$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$

Poles=
stable?



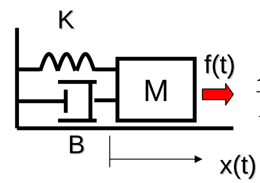

$$\frac{X(s)}{F(s)} = \frac{1}{s^2+K}$$

Poles=
stable?




$$\frac{X(s)}{F(s)} = \frac{1}{s^2+Bs}$$

Poles=
stable?




$$\frac{X(s)}{F(s)} = \frac{1}{s^2+Bs+K}$$

Poles=
stable?



# Examples

| $G(s)$ | **Stable/marginally stable /unstable** |
|---|---|
| $\frac{5(s+2)}{(s+1)(s^2+s+1)}$ | **?** |
| $\frac{5(-s+2)}{(s+1)(s^2+s+1)}$ | **?** |
| $\frac{5}{(s-2)(s^2+3)}$ | **?** |
| $\frac{s^2+3}{(s+1)(s^2-s+1)}$ | **?** |
| $\frac{1}{(s+1)(s^2+1)^2}$ | **?** |
| $\frac{1}{(s^2-1)(s+1)}$ | **???** |



# Summary and Exercises

- Stability for LTI systems
  - (BIBO and asymptotically) stable, marginally stable, unstable
  - Stability for G(s) is determined by poles of G.
- Next
  - Routh-Hurwitz stability criterion to determine stability without explicitly computing the poles of a system.
- Exercises
  - Read Sections 5-1, 5-2, 5-5.
  - Solve examples in the previous slide.