ME451: Control Systems

Lecture 17
Root locus: Examples

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Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ electromechanical
- ✓ Block diagrams
- ✓ Linearization

Analysis

- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot
- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist

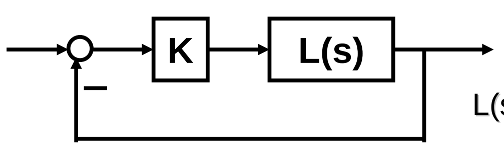
Design

- Design specs
- ➤ Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations &) laboratories

What is Root Locus? (Review)

- Consider a feedback system that has one parameter (gain) K>0 to be designed.



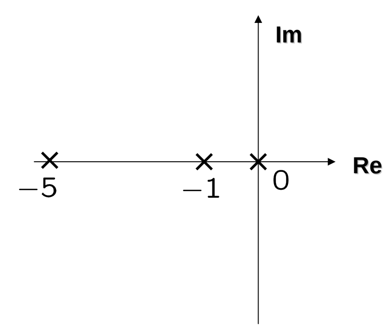
L(s): open-loop TF

- *Root locus* graphically shows how poles of the closed-loop system varies as K varies from 0 to infinity.

Root locus: Step 0 (Mark pole/zero)

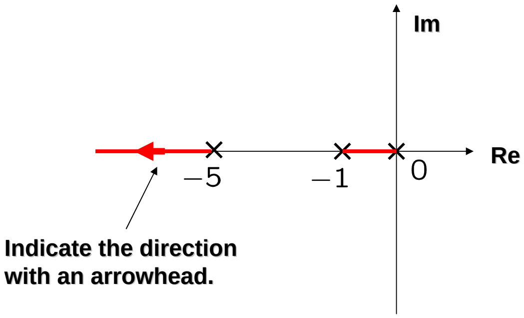
- *Root locus is symmetric w.r.t. the real axis.*
- *The number of branches = order of L(s)*
- *Mark poles of L with "x" and zeros of L with "o".*

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



Root locus: Step 1 (Real axis)

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*
- *RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.*



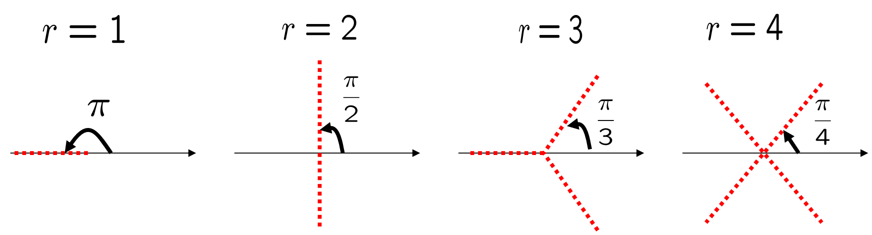
Root locus: Step 2 (Asymptotes)

- *Number of asymptotes = relative degree (r) of L:*

$$r := \deg(\text{den}) - \deg(\text{num})$$

- *Angles of asymptotes are*

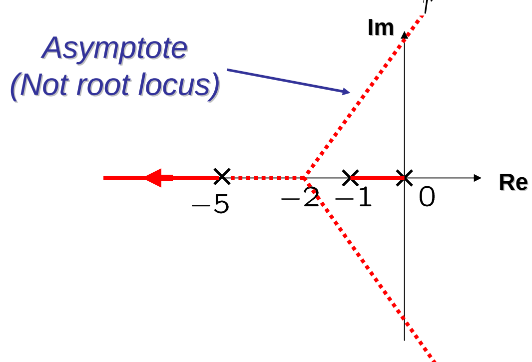
$$\frac{\pi}{r} \times (2k+1), \;\; k = 0, 1, \ldots$$



Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{0 + (-1) + (-5)}{3} = -2$$



Root locus: Step 3 (Breakaway)

- *Breakaway points are among roots of* $\boxed{\dfrac{dL(s)}{ds} = 0}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \frac{dL(s)}{ds} = -\frac{3s^2 + 12s + 5}{(*)} = 0$$

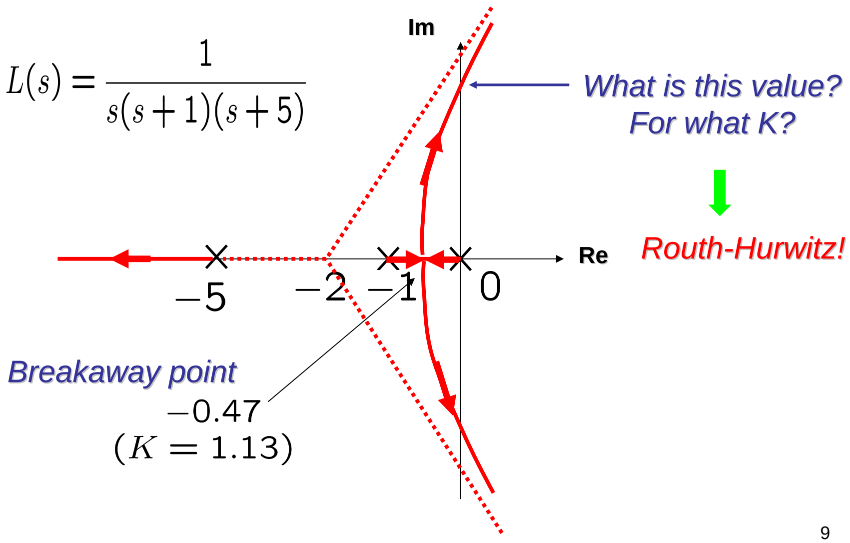
$$\Rightarrow s = -2 \pm \frac{\sqrt{21}}{3}$$

For each candidate s, check the positivity of $K = -\dfrac{1}{L(s)}$

$$\Rightarrow \begin{cases} s = -2 + \dfrac{\sqrt{21}}{3} \approx -0.47 & K \approx 1.13 \\ s = -2 - \dfrac{\sqrt{21}}{3} \approx -3.52 & K \approx -13.1 \end{cases}$$

Root locus: Step 3 (Breakaway)

$$L(s)=\frac{1}{s(s+1)(s+5)}$$



Finding K for critical stability

- Characteristic equation

$$1+\frac{K}{s(s+1)(s+5)}=0 \Leftrightarrow s^3+6s^2+5s+K=0$$

- Routh array

s^3	1	5
s^2	6	K
s^1	$\frac{30-K}{6}$	
s^0	K	

Stability condition

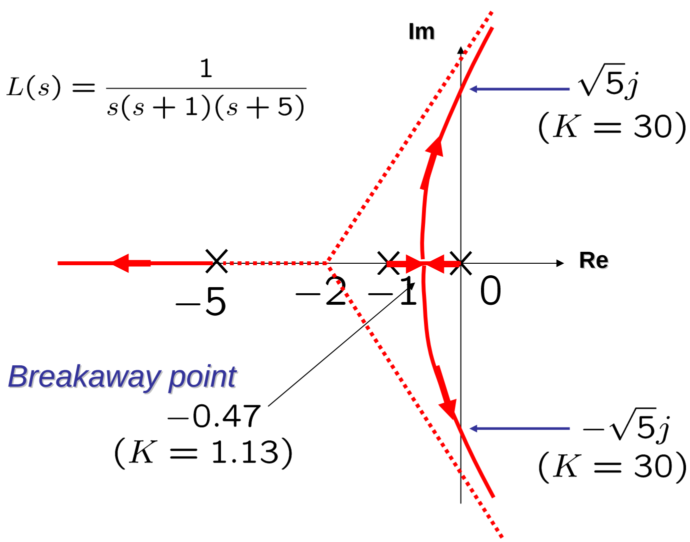
$0<K<30$

- When K=30

$$6s^2+30=0 \Rightarrow s=\pm\sqrt{5}j$$

Root locus

$$L(s)=\frac{1}{s(s+1)(s+5)}$$

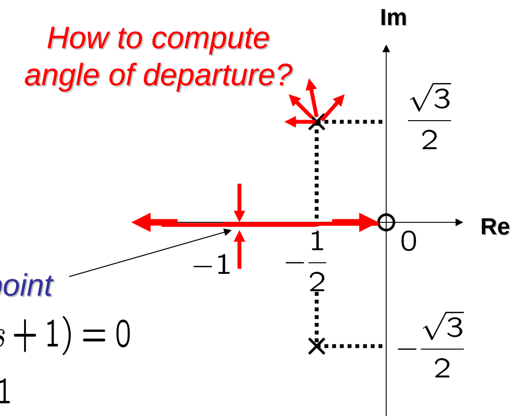


Example with complex poles

$$L(s)=\frac{s}{s^2+s+1}$$

After Steps 0,1,2,3, we obtain

zero	0
pole	$-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$



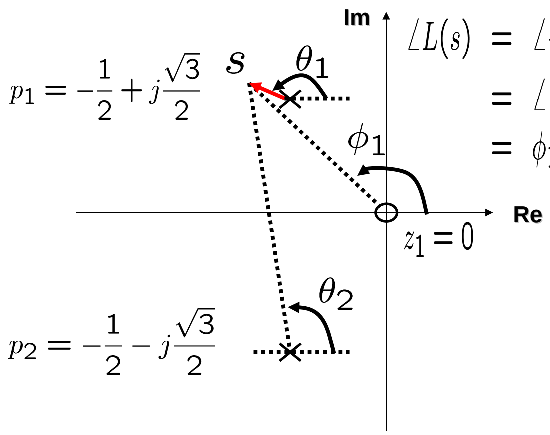
Breakaway point

$$s^2+s+1-s(2s+1)=0$$

$$\Rightarrow s=\pm 1$$

Root locus: Step 4
Angle of departure

- *Angle condition:* For s to be on RL,



$$\begin{aligned}\angle L(s) &= \angle \frac{s-z_1}{(s-p_1)(s-p_2)} \\ &= \angle(s-z_1) - \angle(s-p_1) - \angle(s-p_2) \\ &= \phi_1 - \theta_1 - \theta_2 = 180\end{aligned}$$

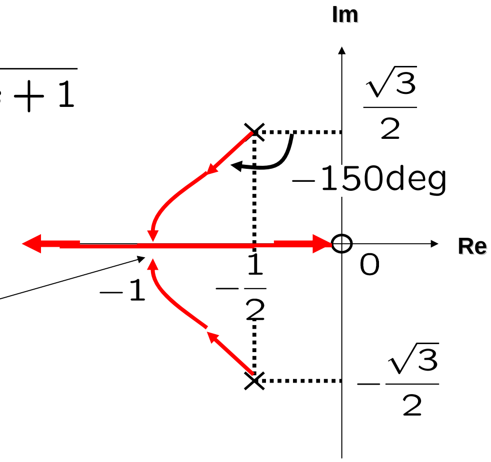
If s is close to p_1

$$\phi_1 \approx 120, \ \theta_2 \approx 90$$

$$\Rightarrow \theta_1 = -150$$

Root locus

$$L(s) = \frac{s}{s^2+s+1}$$



Breakaway point

$s = -1 \ (K = 1)$

Summary and exercises

- Examples for root locus.
  - Gain computation for marginal stability, by using Routh-Hurwitz criterion
  - Angle of departure (Angle of arrival can be obtained by a similar argument.)
- Next, sketch of proofs for root locus algorithm
- Exercises
  - Draw root locus for K>0 (no need to consider K<0) for open-loop transfer functions in
    - Problems 7.5 and 7.7.

Exercises 1

$$L(s) = \frac{s}{s^2+s+1} \qquad L(s) = \frac{1}{(s-1)(s+2)(s+3)}$$

$$L(s) = \frac{s+1}{s^2} \qquad L(s) = \frac{s}{(s+1)(s^2+1)}$$

Exercises 2

$$L(s) = \frac{1}{s(s+1)(s+2)} \qquad L(s) = \frac{1}{(s+1)(s^2+2s+2)}$$

$$L(s) = \frac{1}{s(s+2)(s^2+2s+2)} \qquad L(s) = \frac{1}{(s^2+4s+5)(s^2+2s+5)}$$

Exercises 3

$$L(s) = \frac{1}{s(s+3)(s^2+2s+2)} \qquad L(s) = \frac{1}{s(s+1)(s+2)(s^2+2s+2)}$$

$$L(s) = \frac{s+1}{s^2+4s+5} \qquad L(s) = \frac{s+3}{(s+1)(s^2+4s+5)}$$

Exercises 4

$$L(s) = \frac{s+4}{(s+1)(s+3)(s^2+4s+5)} \qquad L(s) = \frac{s+2}{(s^2+2s+5)(s^2+6s+10)}$$

$$L(s) = \frac{s^2+2s+2}{s(s+2)(s+3)} \qquad L(s) = \frac{(s+2)(s+3)}{s(s+1)}$$