ME451: Control Systems

Lecture 10
Routh-Hurwitz stability criterion

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Course roadmap

Modeling
- ✔ Laplace transform
- ✔ Transfer function
- ✔ Models for systems
  - ✔ electrical
  - ✔ mechanical
  - ✔ electromechanical
- ✔ Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- ✔ Stability
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

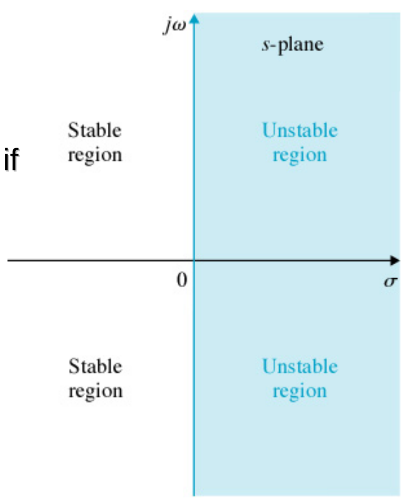
(Matlab simulations &) laboratories

Stability summary (review)

Let s_i be poles of rational G. Then, G is …

- (BIBO, asymptotically) stable if $Re(s_i)<0$ for all i.
- marginally stable if
  - $Re(s_i)<=0$ for all i, and
  - simple root for $Re(s_i)=0$
- unstable if

  it is neither stable nor marginally stable.



Routh-Hurwitz criterion

- This is for LTI systems with a *polynomial* denominator (without sin, cos, exponential etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots.
- No proof is provided in any control textbook.

Polynomial and an assumption

- Consider a polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

- Assume $a_0 \neq 0$
  - If this assumption does not hold, Q can be factored as

$$Q(s) = s^m \underbrace{\left(\hat{a}_{n-m} s^{n-m} + \cdots + \hat{a}_1 s + \hat{a}_0\right)}_{\hat{Q}(s)}$$

  where $\hat{a}_0 \neq 0$
  - The following method applies to the polynomial $\hat{Q}(s)$

Routh array

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

From the given polynomial

Routh array (How to compute the third row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$\vdots$$

Routh array (How to compute the fourth row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

$$\vdots$$

Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.

Example 1

$$Q(s) = s^3 + s^2 + 2s + 8 \; (= (s+2)(s^2 - s + 4))$$

Routh array

$$
\begin{array}{l|ll}
s^3 & 1 & 2 \\
s^2 & 1 & 8 \\
s^1 & -6 & \\
s^0 & 8 &
\end{array}
$$

$$\frac{2-8}{1} \qquad \frac{8 \times (-6) - 0}{-6}$$

Two sign changes in the first column $1 \to -6 \to 8$ ⇒ Two roots in RHP $\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$

Example 2

$$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh array

$$
\begin{array}{l|lll}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & \not{0}\;\varepsilon & 6 & \\
s^2 & \frac{4\varepsilon - 12}{\varepsilon} & 10 & \\
s^1 & 6 & & \\
s^0 & 10 & &
\end{array}
$$

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number. In this case, Q has some roots in RHP.

Two sign changes in the first column ⇒ Two roots in RHP

$$\varepsilon \to \underbrace{\frac{4\varepsilon - 12}{\varepsilon}}_{<0} \to 6$$

Example 3

$$Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

Routh array

$$
\begin{array}{l|lll}
s^4 & 1 & 3 & 2 \\
s^3 & 1 & 2 & \\
s^2 & 1 & 2 & \\
s^1 & \not{0}\;2 & & \\
s^0 & 2 & &
\end{array}
$$

If zero row appears in Routh array, Q has roots either on the imaginary axis or in RHP.

No sign changes in the first column ⇒ No roots in RHP

But some roots are on imag. axis.

Take derivative of an *auxiliary polynomial* (which is a factor of Q(s)) $s^2 + 2$

Example 4

$$Q(s) = s^3 + 3Ks^2 + (K+2)s + 4$$

Find the range of K s.t. Q(s) has all roots in the left half plane. (Here, K is a design parameter.)

Routh array

s^3	1	$K+2$
s^2	$3K$	4
s^1	$\frac{3K(K+2)-4}{3K}$	
s^0	4	

No sign changes in the first column

$$\Rightarrow \begin{cases} 3K > 0 \\ 3K(K+2) - 4 > 0 \end{cases}$$

$$\Rightarrow K > -1 + \frac{\sqrt{21}}{3}$$

Simple & important criteria for stability

- 1st order polynomial $Q(s) = a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_1 and a_0 have the same sign

- 2nd order polynomial $Q(s) = a_2 s^2 + a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_2, a_1 and a_0 have the same sign

- Higher order polynomial $Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$

  All roots are in LHP $\Rightarrow$ All a_k have the same sign

Examples

$Q(s)$	**All roots in open LHP?**
$3s + 5$	**Yes / No**
$-2s^2 - 5s - 100$	**Yes / No**
$523s^2 - 57s + 189$	**Yes / No**
$(s^2 + s - 1)(s^2 + s + 1)$	**Yes / No**
$s^3 + 5s^2 + 10s - 3$	**Yes / No**

Summary and Exercises

- Routh-Hurwitz stability criterion
  - Routh array
  - Routh-Hurwitz criterion is applicable to only polynomials (so, it is not possible to deal with exponential, sin, cos etc.).
- Next,
  - Routh-Hurwitz criterion in control examples
- Exercises
  - Read Section 6.
  - Do Examples and Problems 6-2.