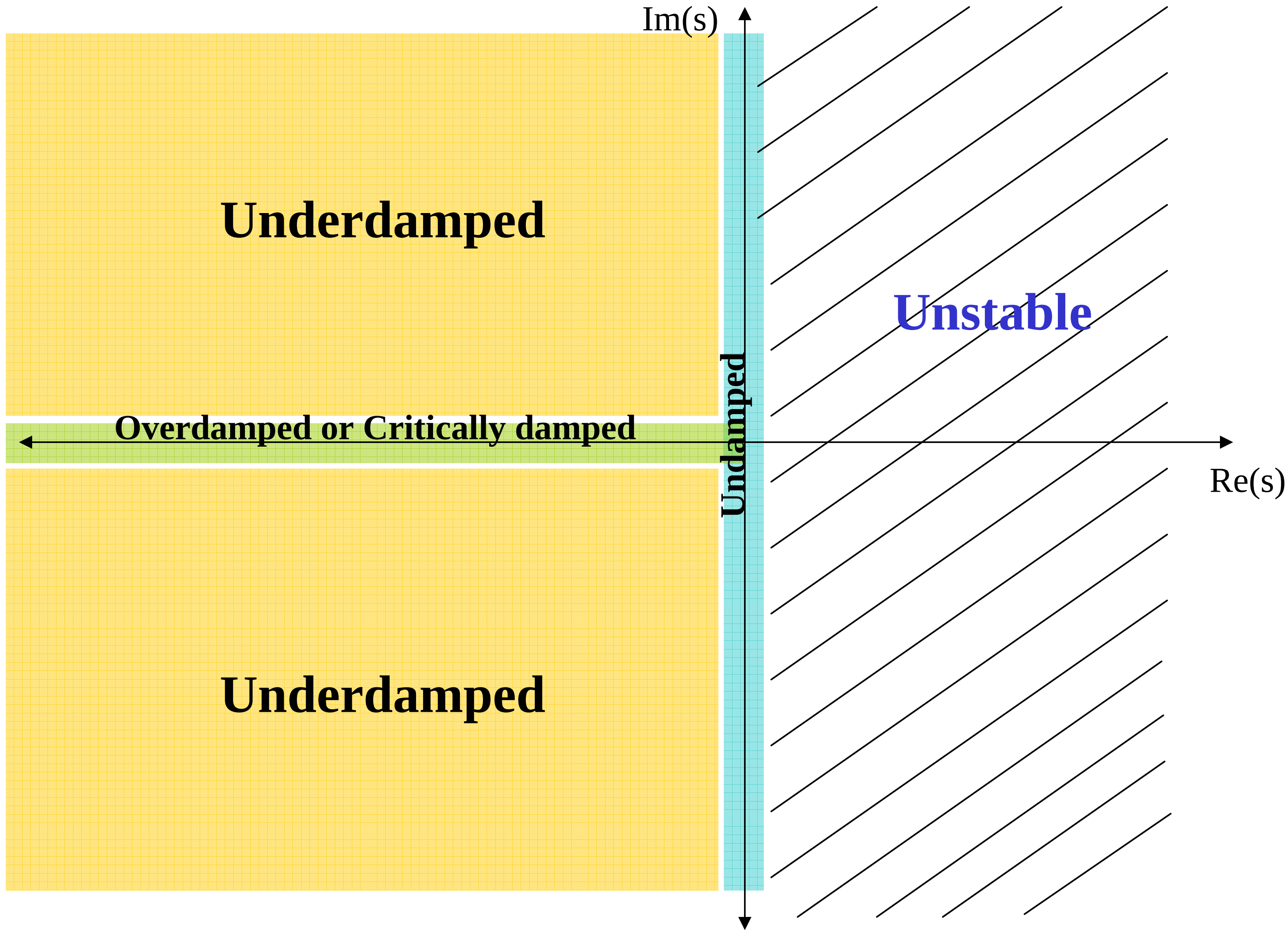
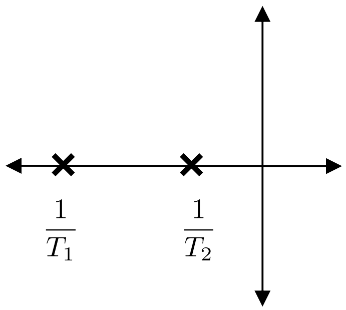
Second order system response.



Overdamped system response

- System transfer function : $H(s) = \dfrac{K}{(T_1 s + 1)(T_2 s + 1)}$



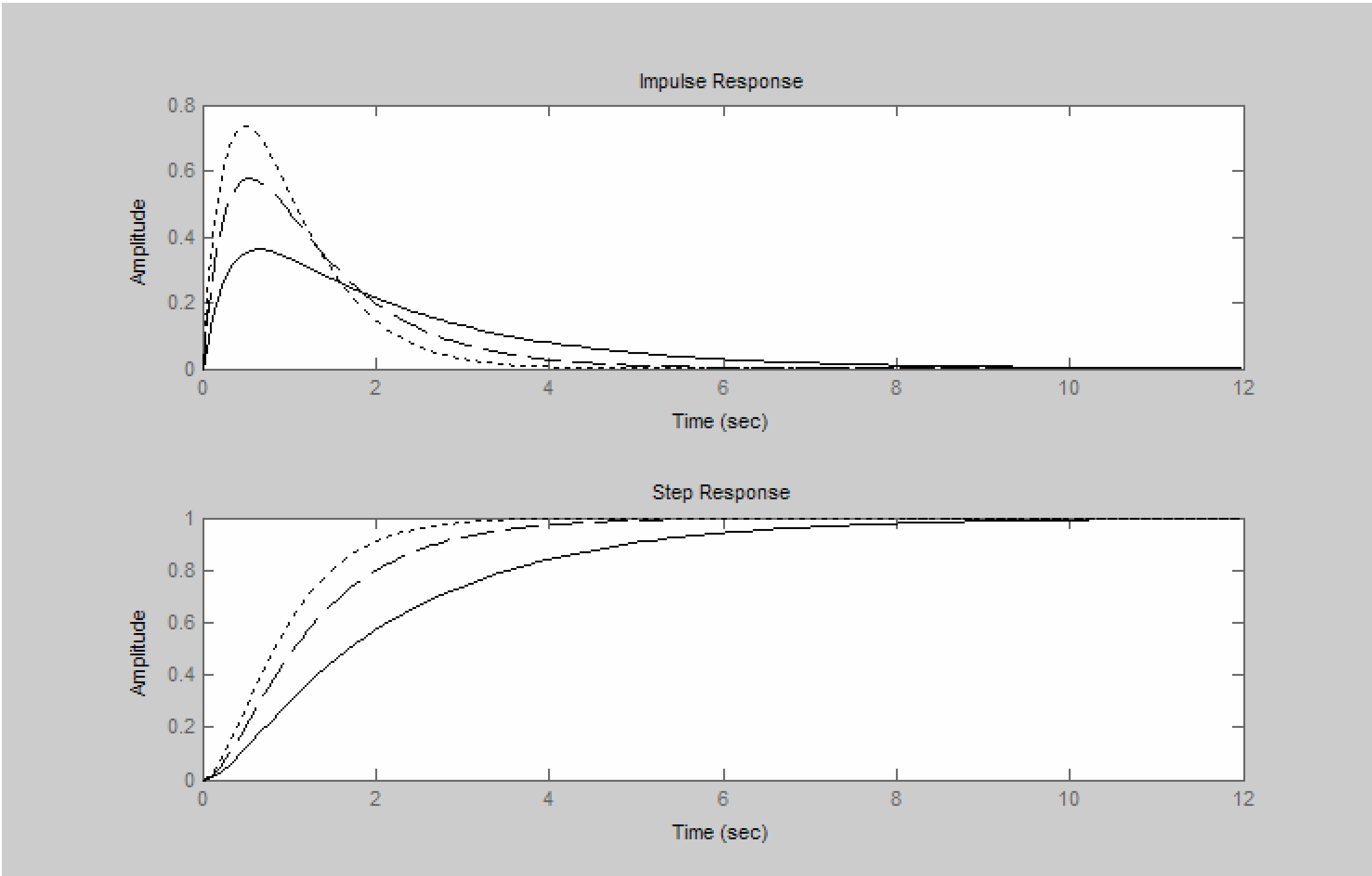
- Impulse response :

$$h(t) \; = \; \mathcal{L}^{-1}[H(s)] \; = \; \frac{K}{T_2 - T_1}\left(e^{-t/T_2} - e^{-t/T_1}\right) 1(t)$$

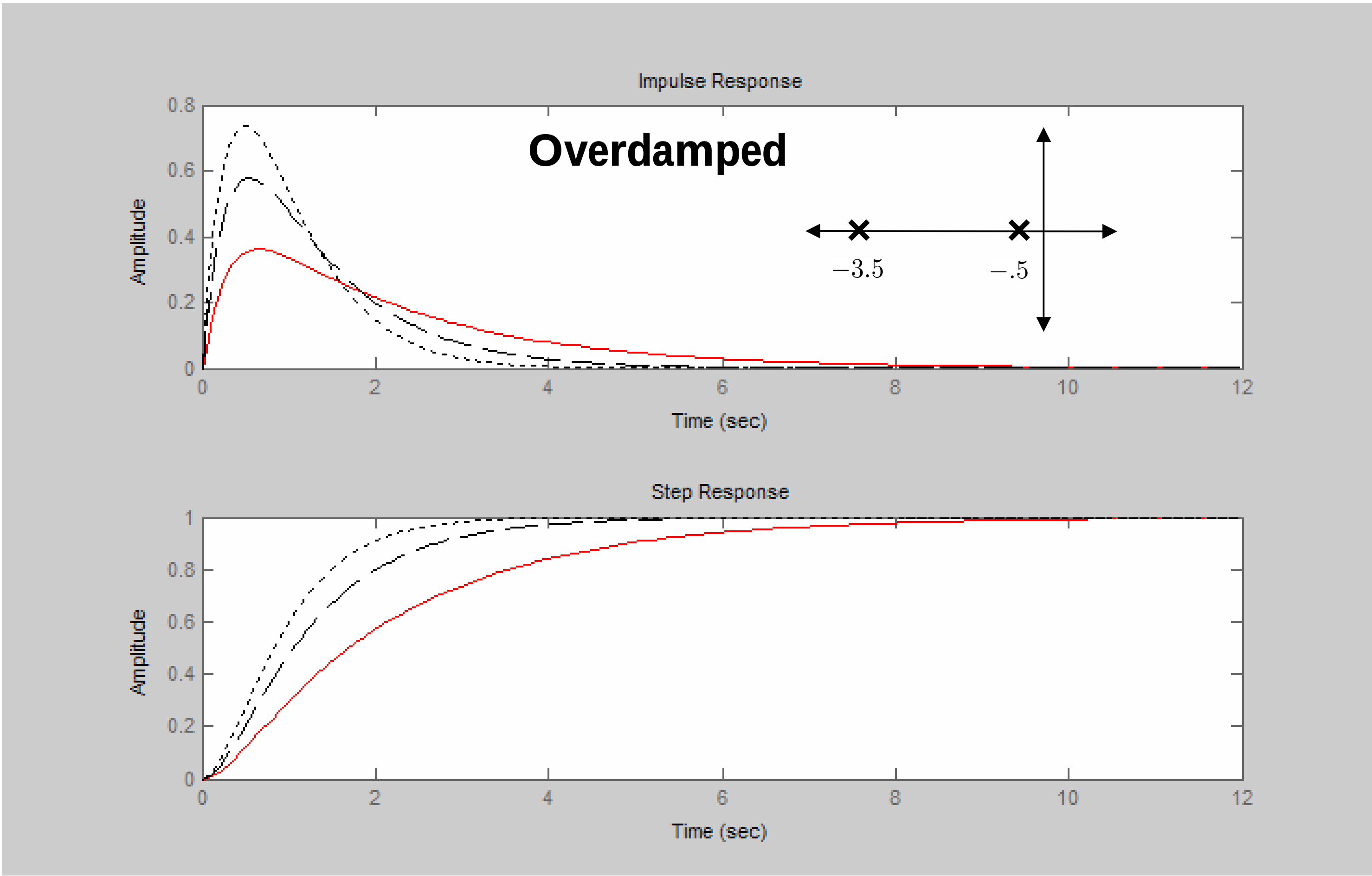
- Step response :

$$y_{step}(t) \; = \; \mathcal{L}^{-1}[H(s)/s] \; = \; K\left(1 + \frac{T_1}{T_2 - T_1} e^{-t/T_2} - \frac{T_2}{T_2 - T_1} e^{-t/T_1}\right) 1(t)$$

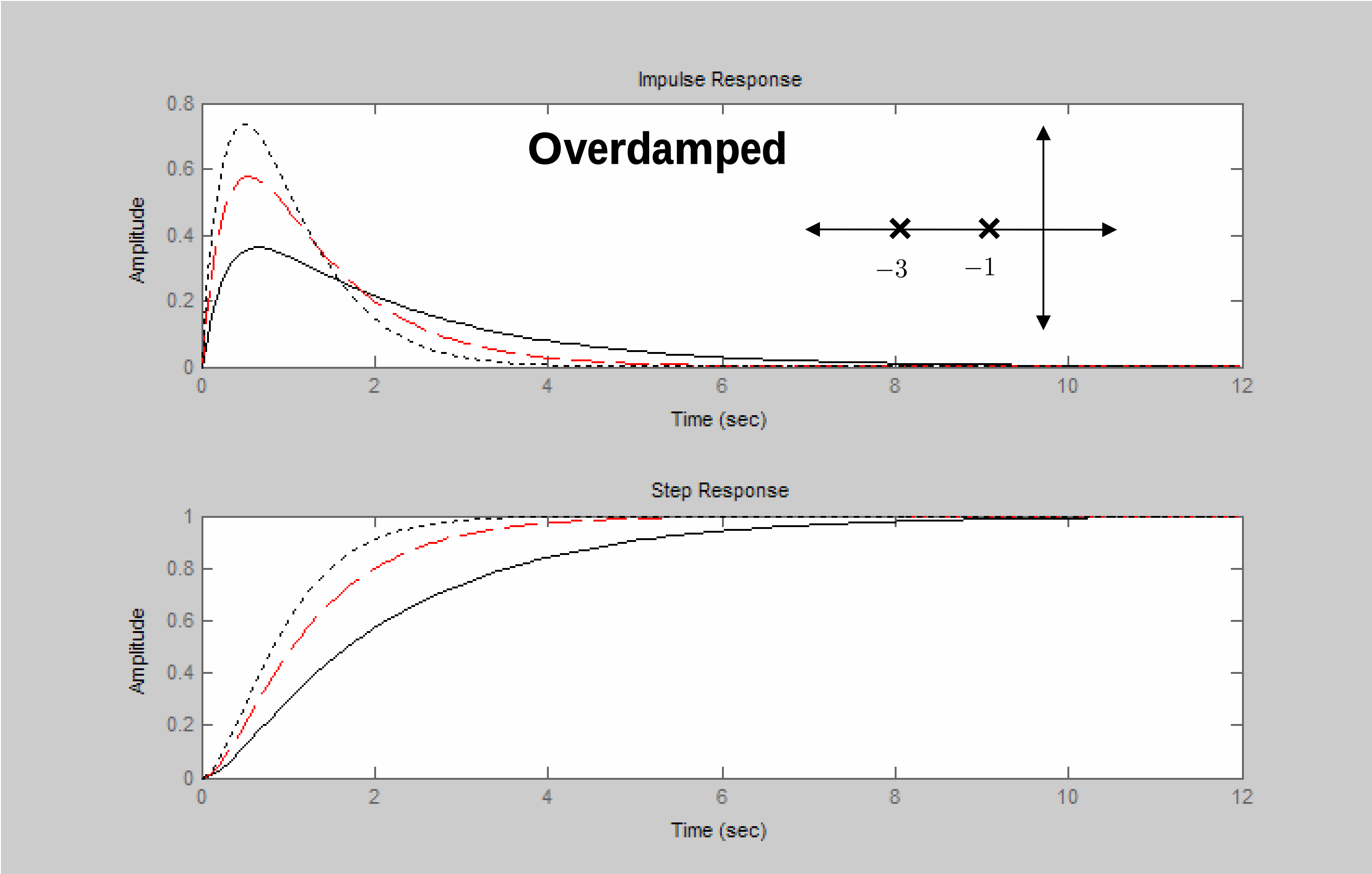
Overdamped and critically damped system response.



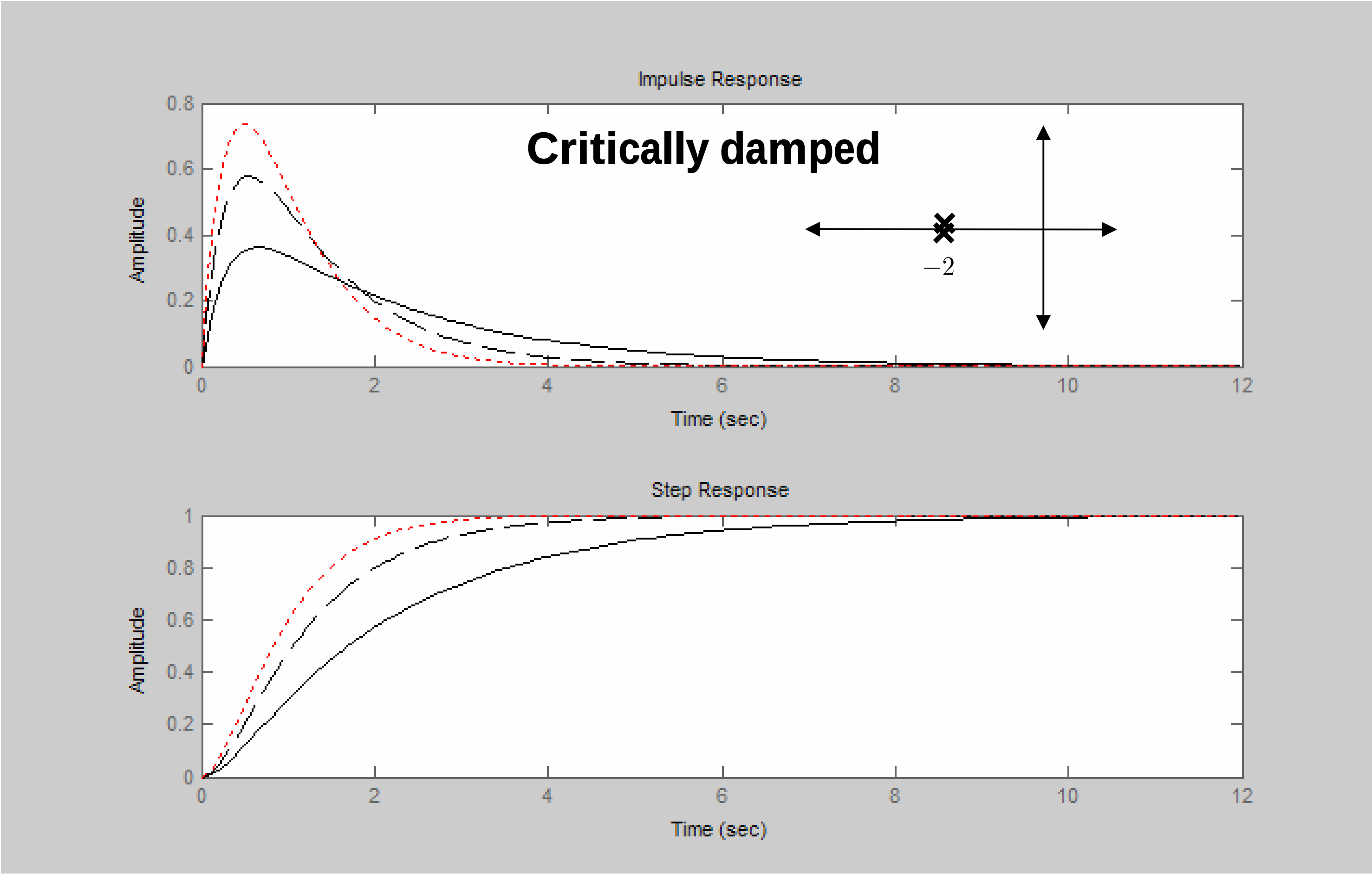
Overdamped and critically damped system response.



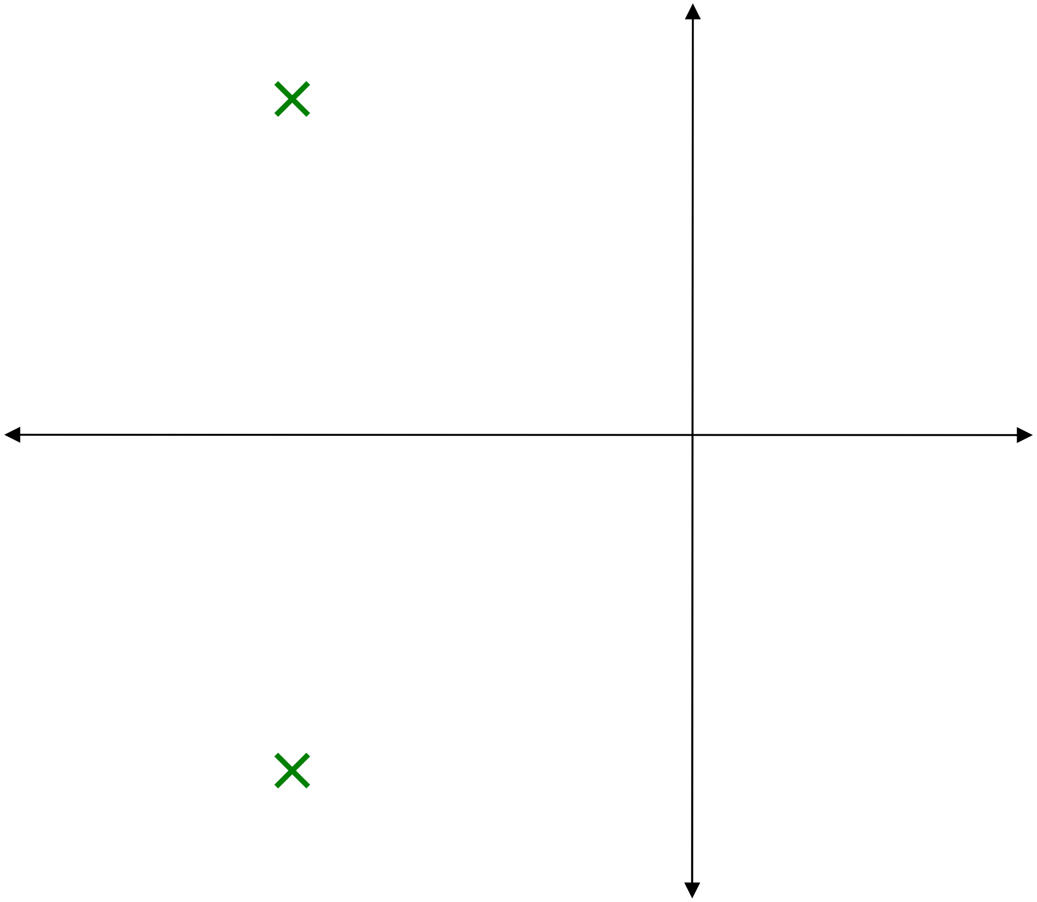
Overdamped and critically damped system response.



Overdamped and critically damped system response.



Polar vs. Cartesian representations.



Polar vs. Cartesian representations.

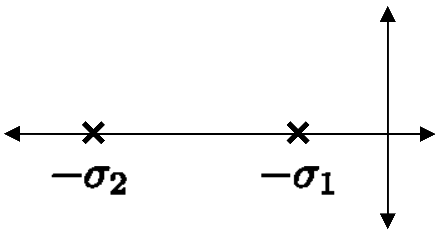
- System transfer function :

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$p_{1,2} = -\sigma \pm \omega\, j$$

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}}_{\text{Polar}} = \underbrace{\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}}_{\text{Cartesian}}$$

All 4 cases **Unless overdamped**



$$H(s) = \frac{\sigma_1\sigma_2}{(s+\sigma_1)(s+\sigma_2)}$$

… Cartesian overdamped

$$p_1 = -\sigma_1\ ,\ p_2 = -\sigma_2$$

- Significance of the damping ratio :

$\zeta > 1$ … Overdamped

$\zeta = 1$ … Critically damped

$1 > \zeta > 0$ … Underdamped

$\zeta = 0$ … Undamped

Polar vs. Cartesian representations.

- System transfer function :

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

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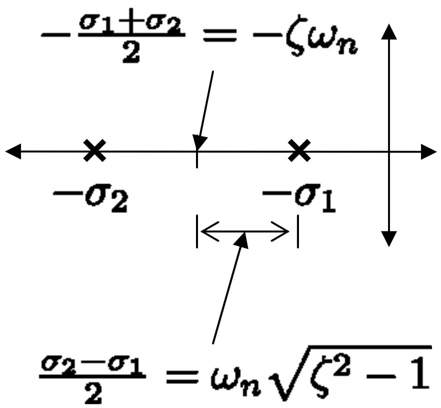
All 4 cases (Polar)

Unless overdamped (Cartesian)



$$H(s) = \frac{\sigma_1\sigma_2}{(s+\sigma_1)(s+\sigma_2)}$$

… Cartesian overdamped

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Polar vs. Cartesian representations.

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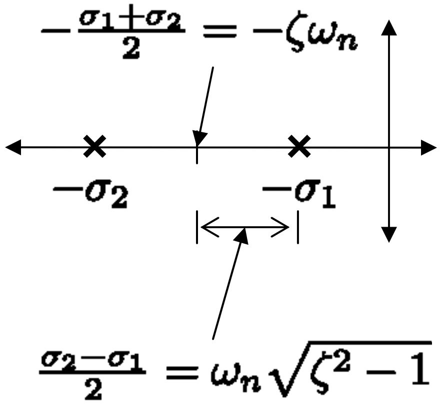
$p_{1,2} = -\sigma \pm \omega\, j$

All 4 cases (Polar)

Unless overdamped (Cartesian)



$$H(s) = \frac{\sigma_1\sigma_2}{(s+\sigma_1)(s+\sigma_2)}$$

… Cartesian overdamped

$p_1 = -\sigma_1\ ,\ p_2 = -\sigma_2$

- Significance of the damping ratio :

$\zeta > 1$	… Overdamped
$\zeta = 1$	… Critically damped
$1 > \zeta > 0$	… Underdamped
$\zeta = 0$	… Undamped

Overdamped case:

$$-\sigma_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\sigma_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$\omega_n = \sqrt{\sigma_1\sigma_2}$$

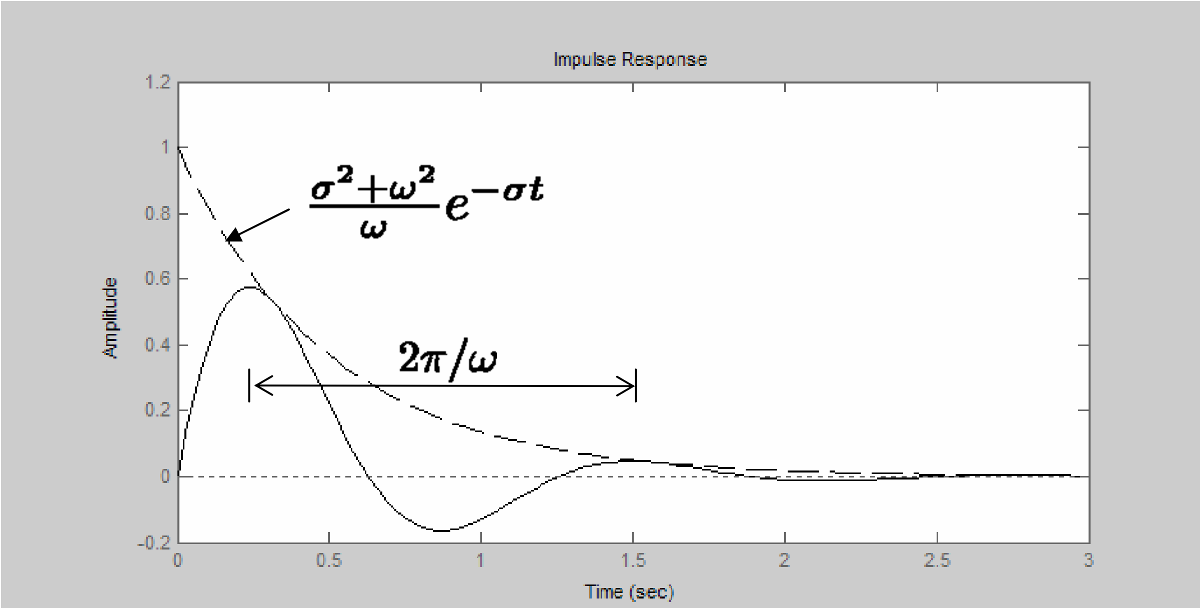
$$\zeta = \frac{\sigma_1 + \sigma_2}{2\sqrt{\sigma_1\sigma_2}}$$

Second order impulse response – Underdamped and Undamped

$$H(s) = \underbrace{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}}_{\text{Polar}} = \underbrace{\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}}_{\text{Cartesian}}$$

- Impulse response :

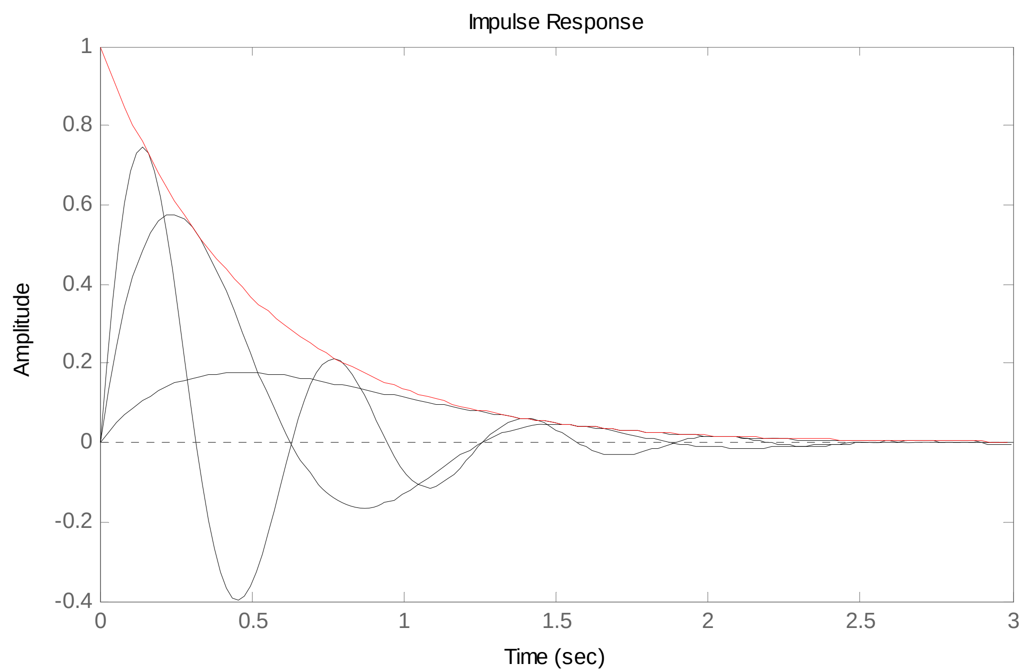
$$h(t) = \mathcal{L}^{-1}\left[\frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}\right] = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$



Second order impulse response – Underdamped and Undamped

Increasing ω / Fixed σ

$$H(s) = \frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}$$

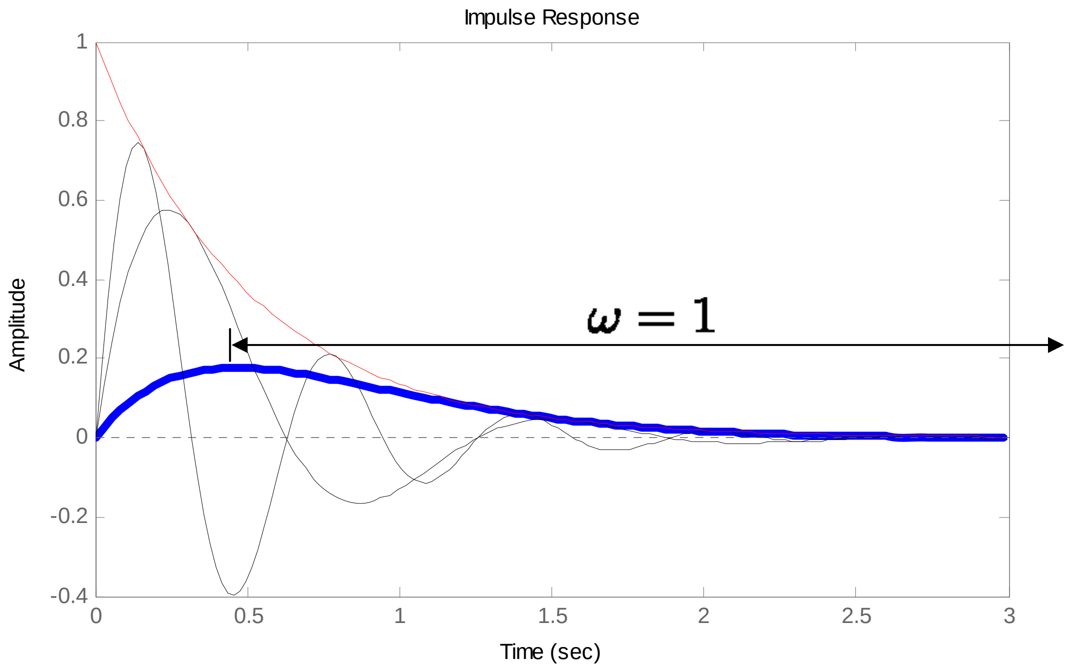


$$h(t) = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$

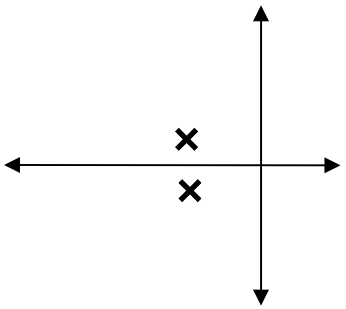
Second order impulse response – Underdamped and Undamped

Increasing ω / Fixed σ

$$H(s) = \frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}$$



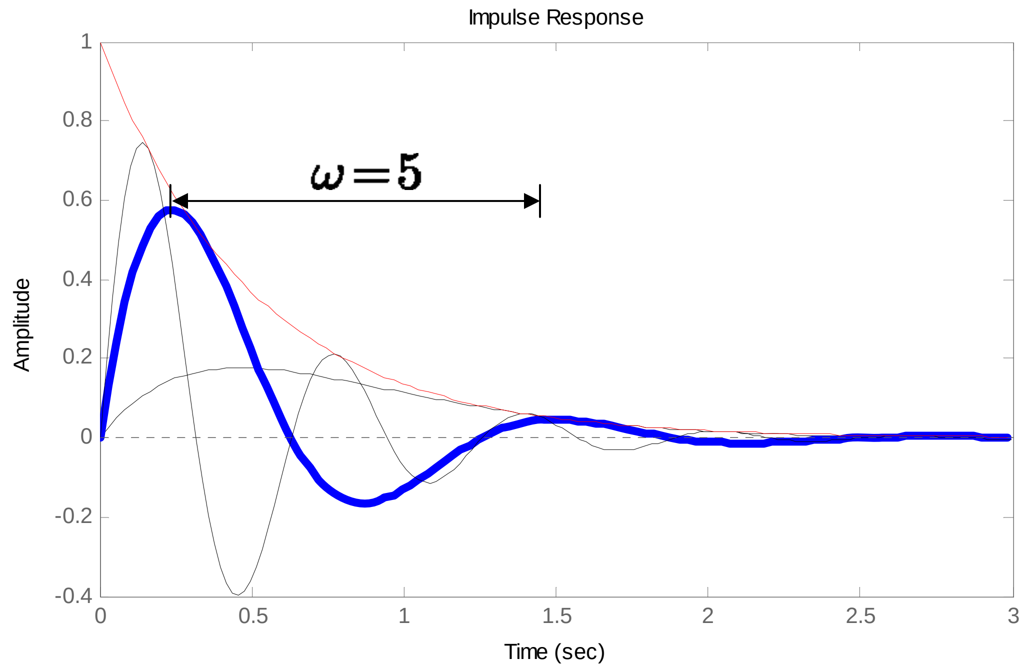
$$h(t) = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$

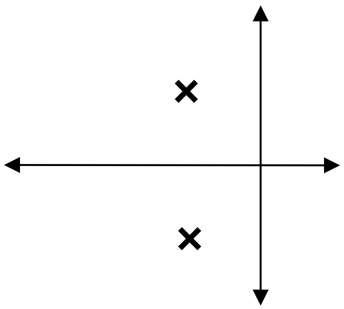


Second order impulse response – Underdamped and Undamped

Increasing ω / Fixed σ

$$H(s) = \frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}$$





$$h(t) = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing ω / Fixed σ

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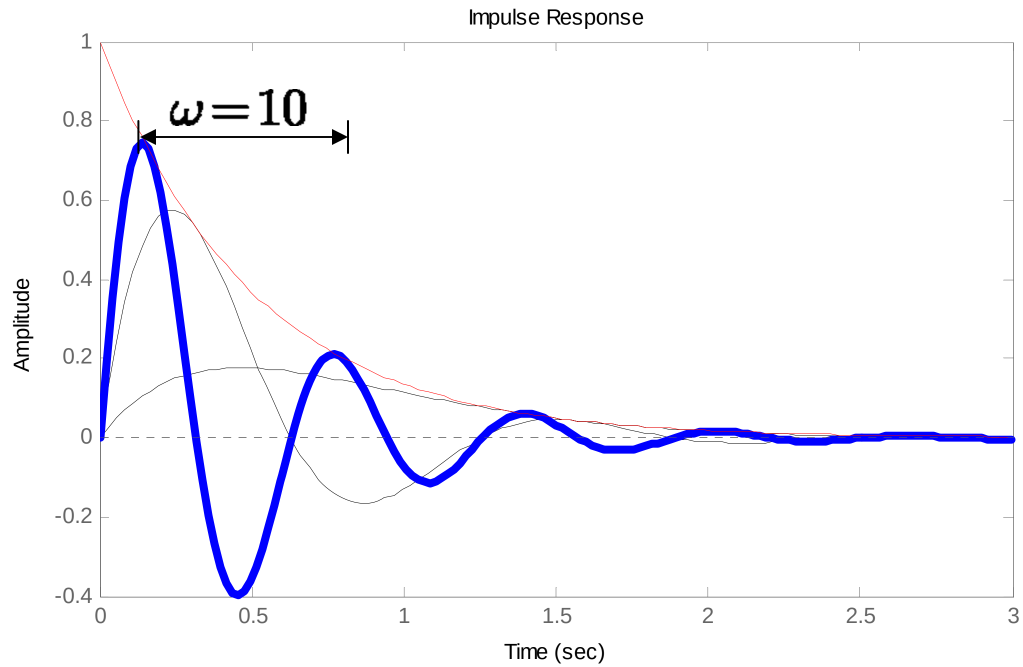
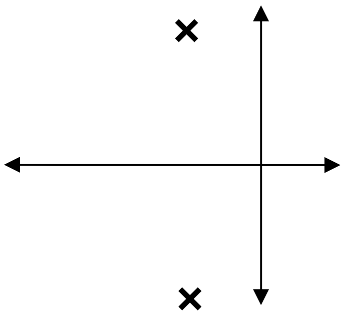


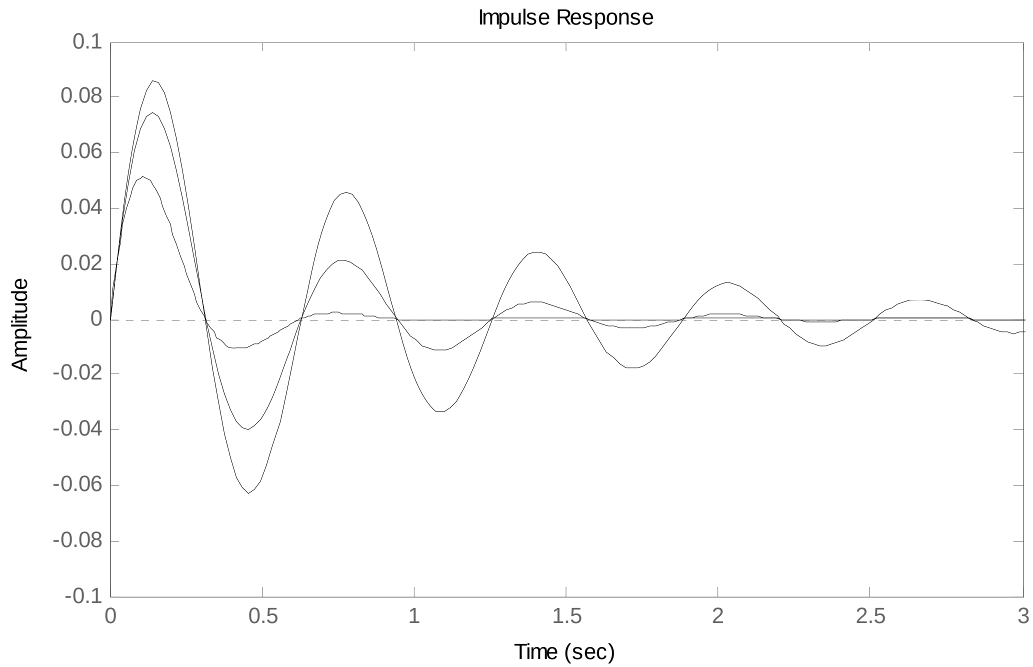


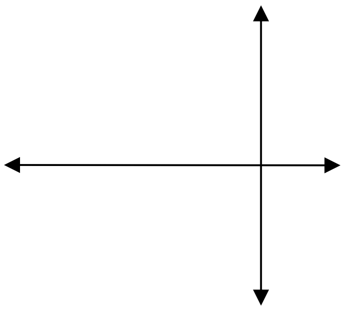
$$h(t) = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing σ / Fixed ω

$$H(s) = \frac{\sigma^2 + \omega^2}{(s+\sigma)^2 + \omega^2}$$





$$h(t) = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\ 1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing σ / Fixed ω

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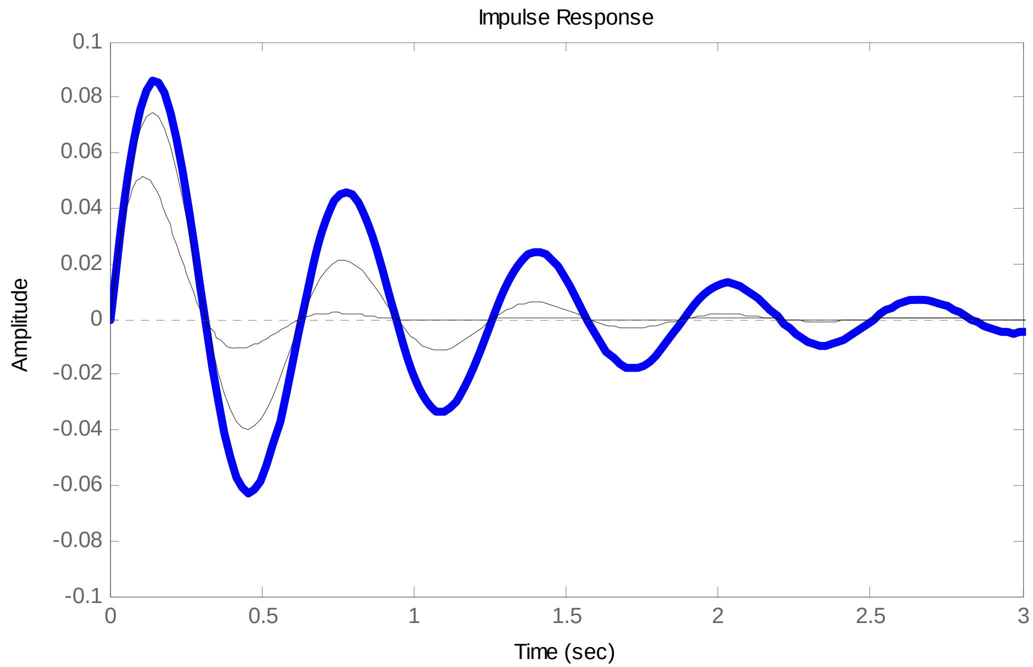

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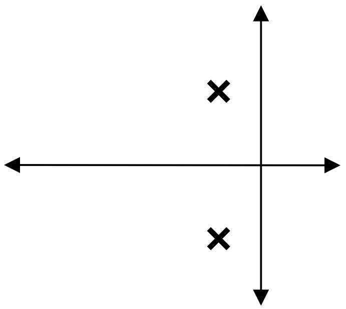

Second order impulse response – Underdamped and Undamped

Increasing σ / Fixed ω

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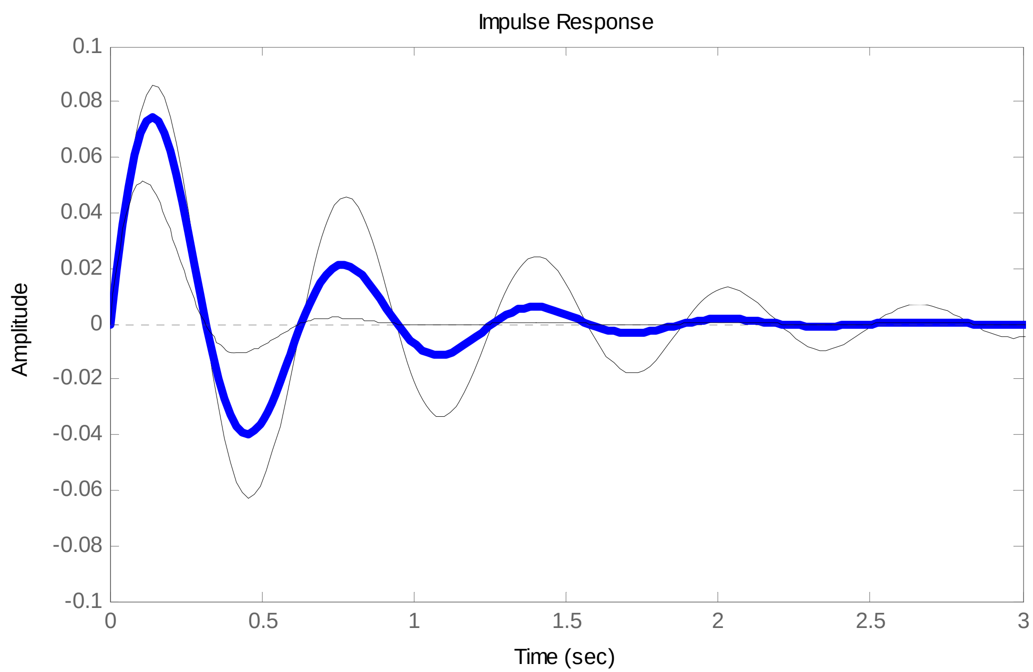
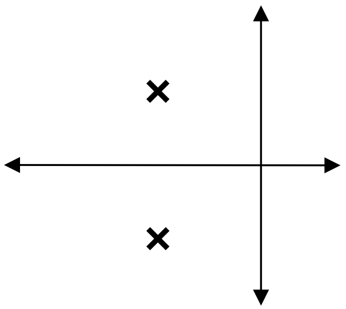


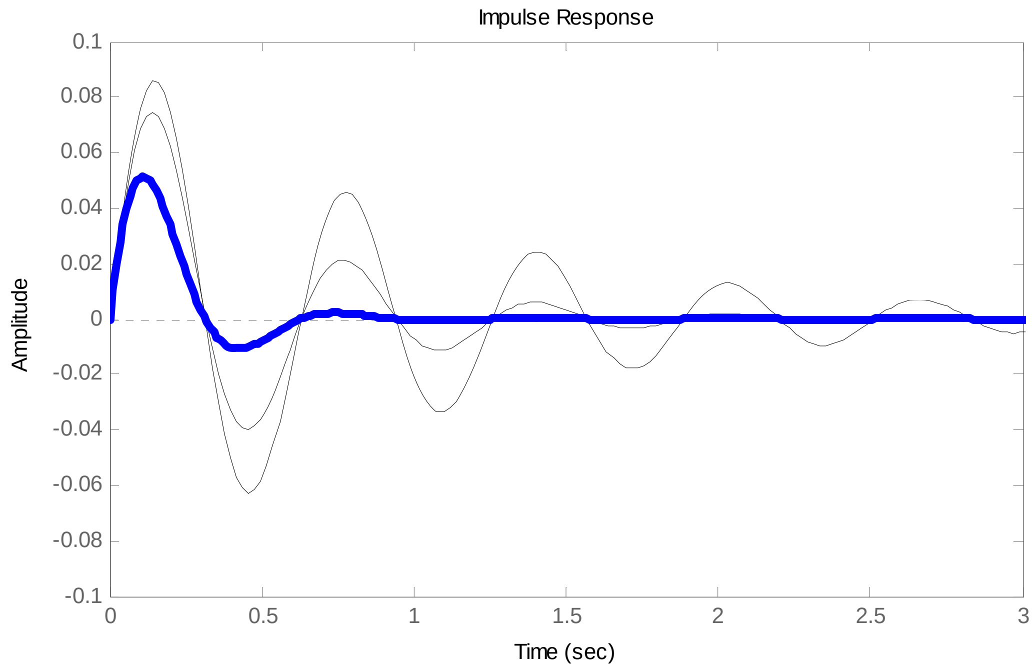


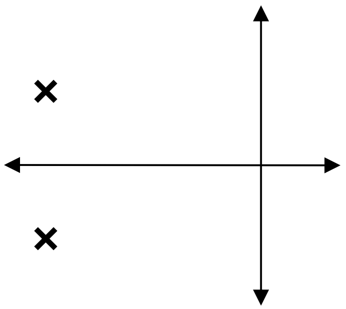
$$h(t) = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t)\, 1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing σ / Fixed ω

$$H(s) = \frac{\sigma^2+\omega^2}{(s+\sigma)^2+\omega^2}$$

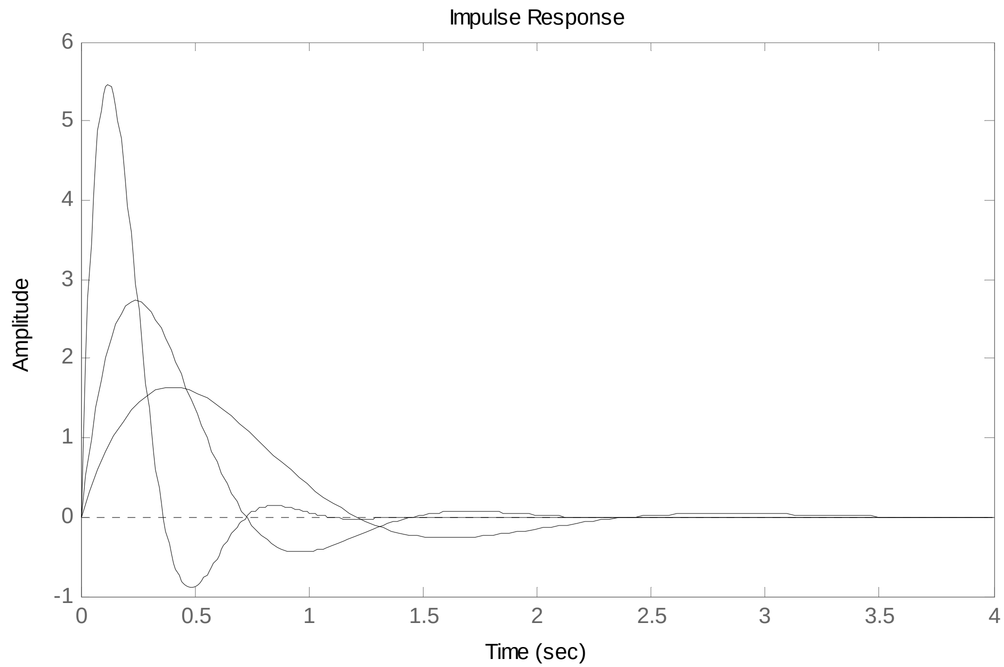




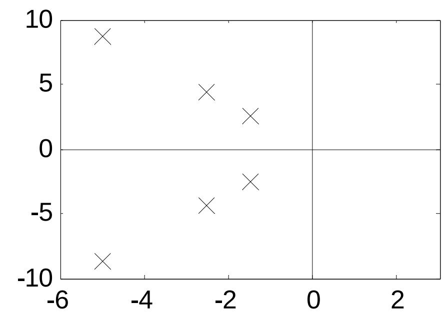
$$h(t) = \frac{\sigma^2+\omega^2}{\omega} e^{-\sigma t}\sin(\omega t)\,1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ

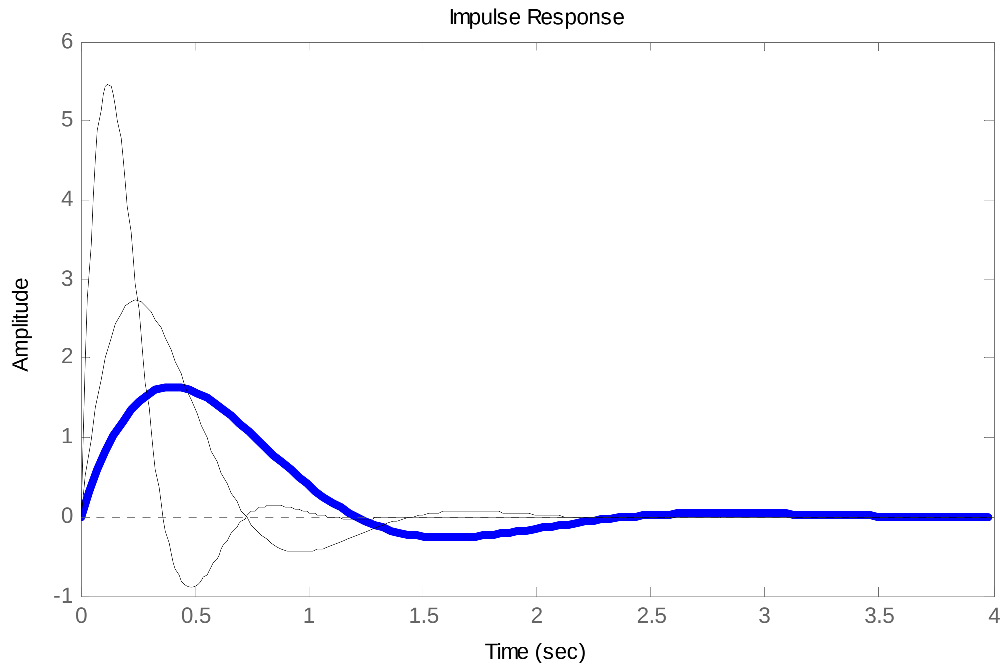


$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$

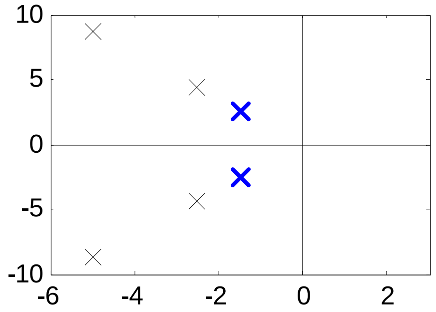


Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ

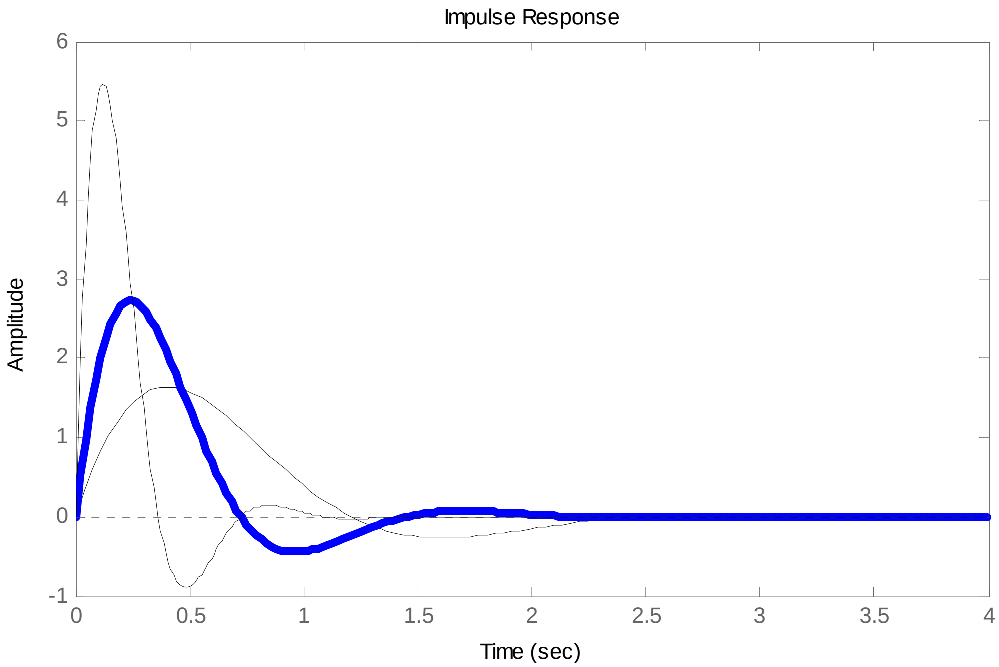


$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$



Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ



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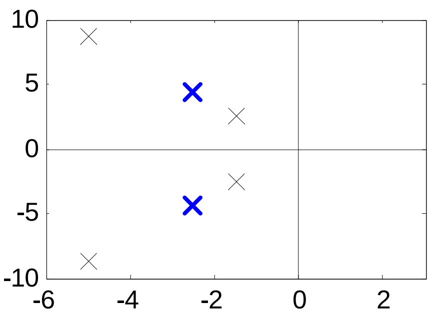
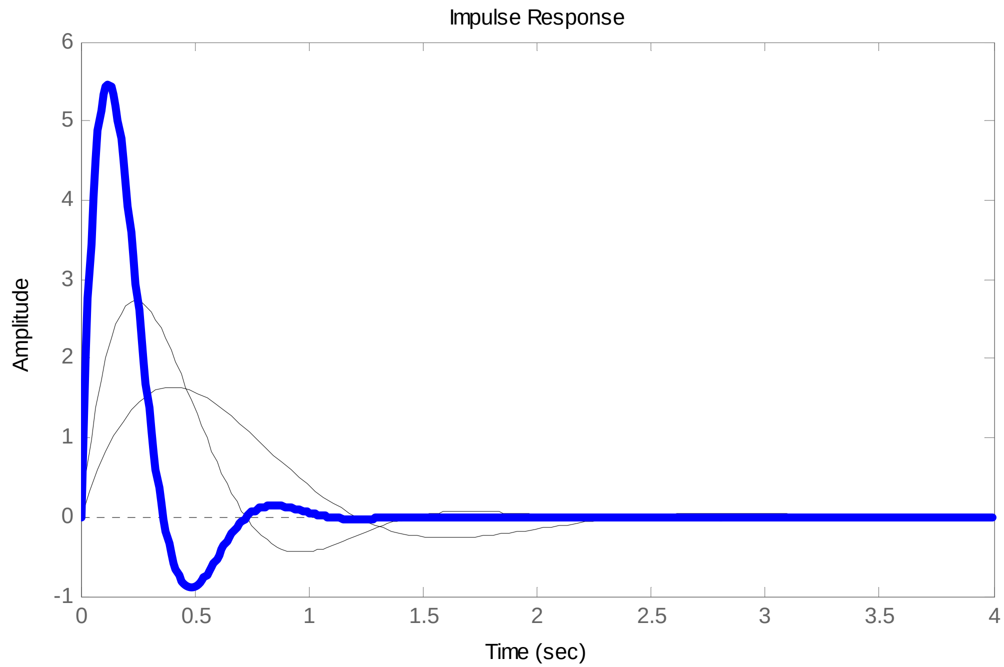
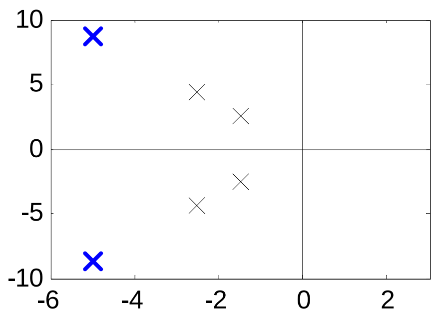


Second order impulse response – Underdamped and Undamped

Increasing ω_n / Fixed ξ

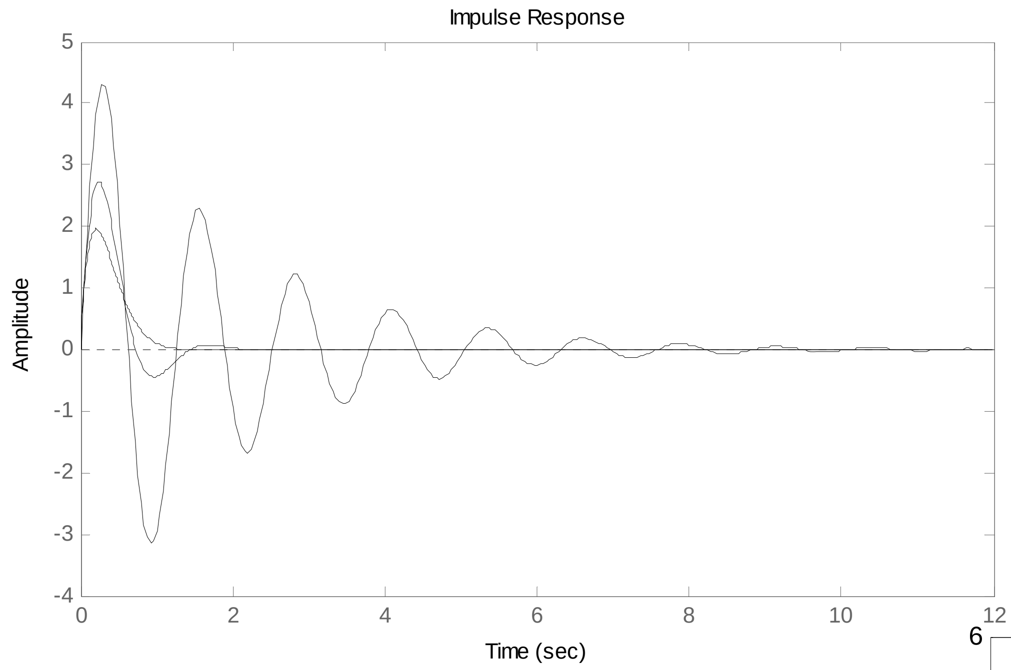


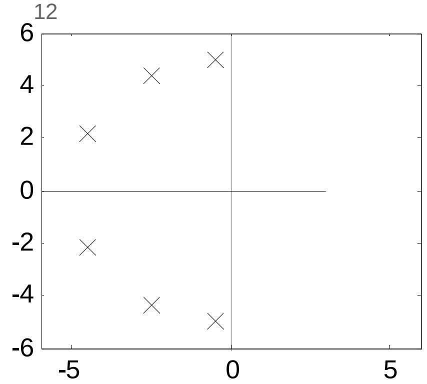
$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$



Second order impulse response – Underdamped and Undamped

Increasing ξ / Fixed ω_n

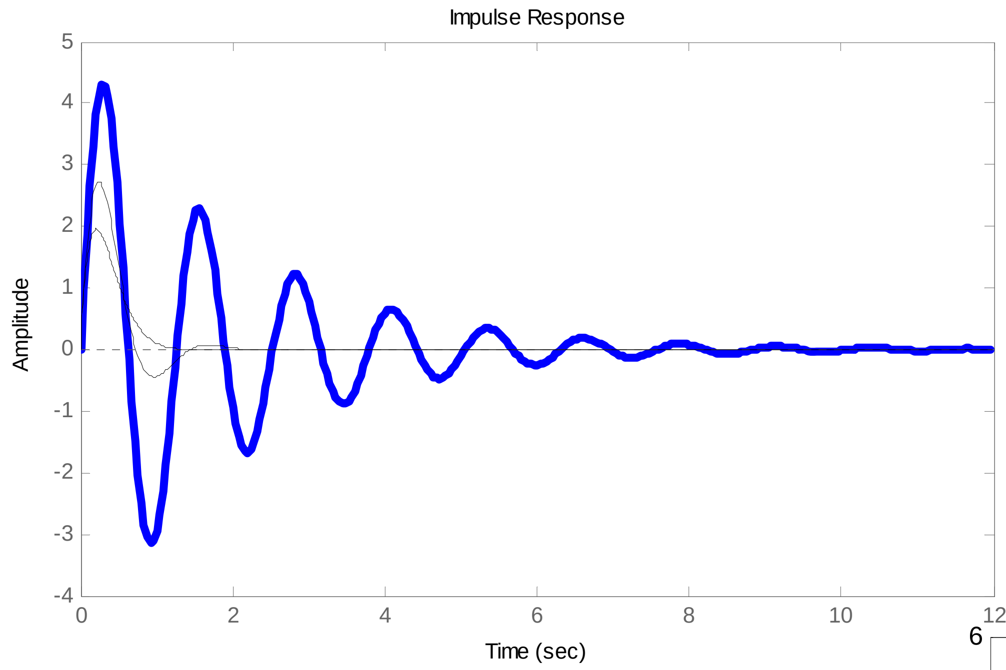


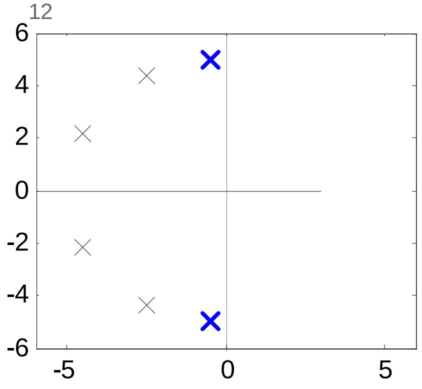


$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing ξ / Fixed ω_n

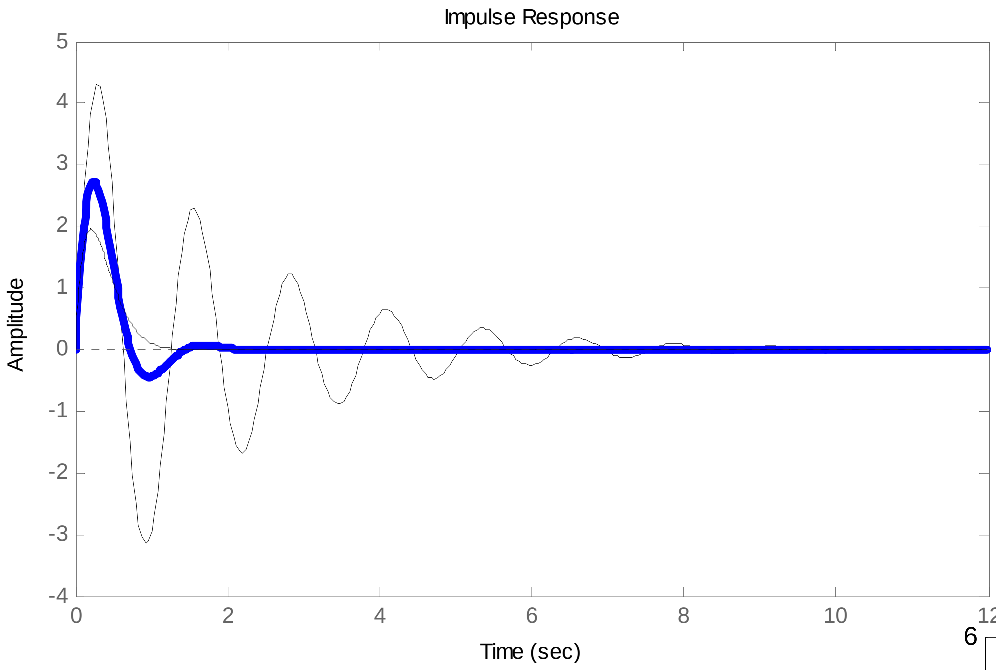




$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$

Second order impulse response – Underdamped and Undamped

Increasing ξ / Fixed ω_n



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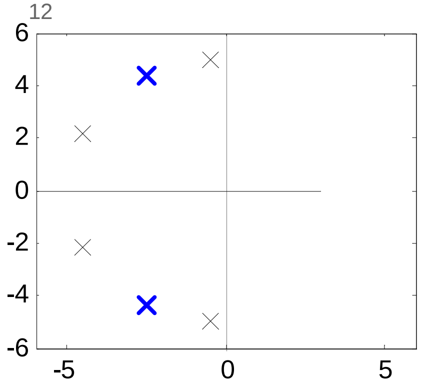
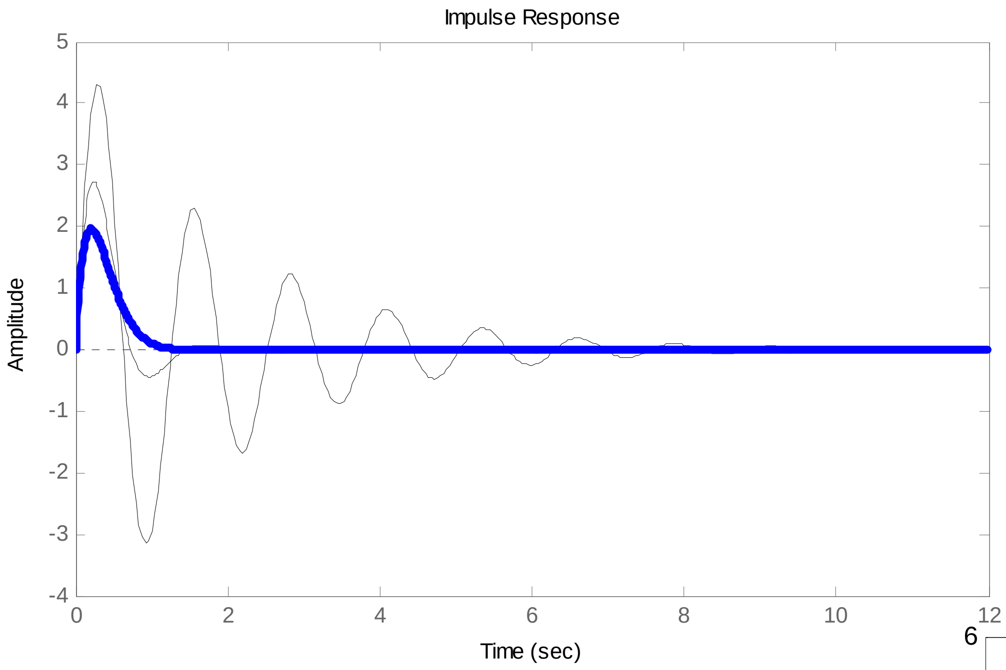
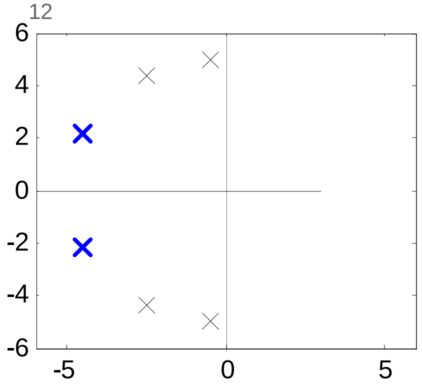


Second order impulse response – Underdamped and Undamped

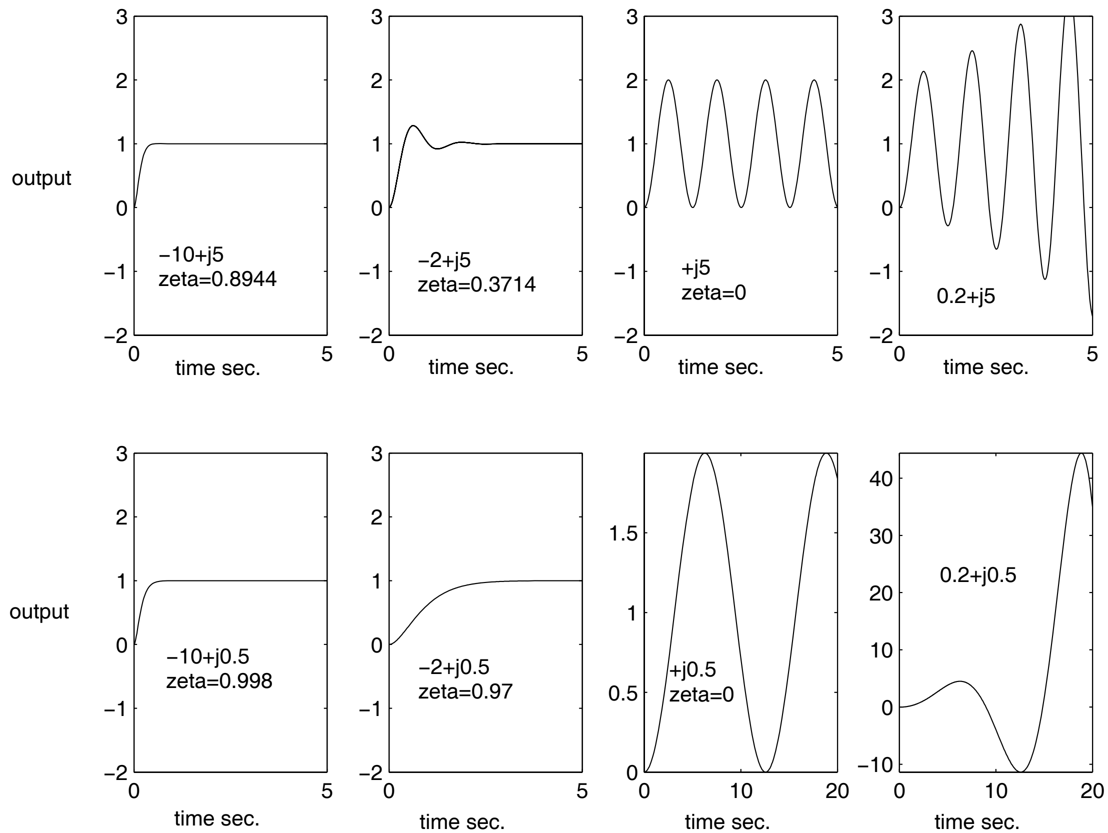
Increasing ξ / Fixed ω_n





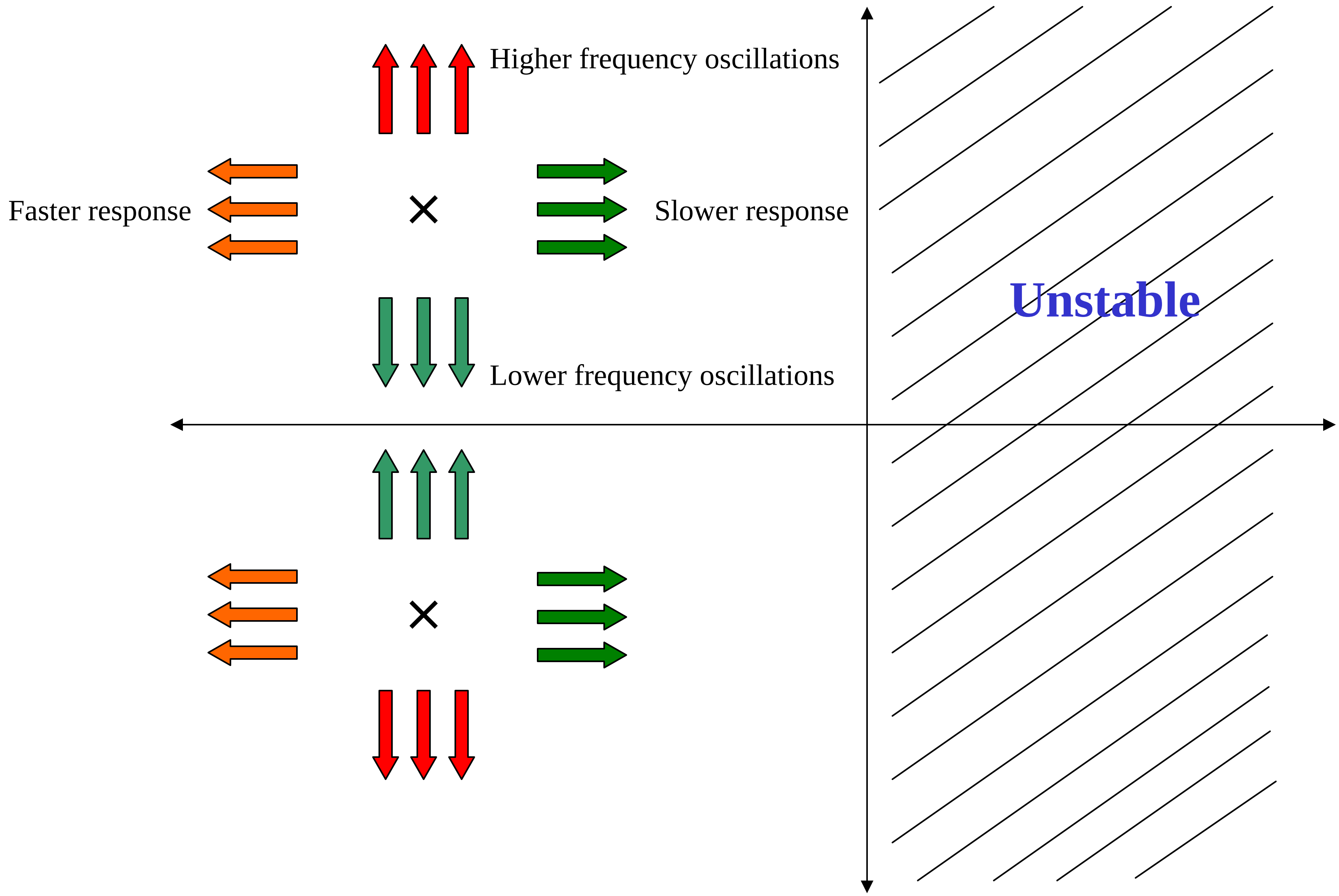
$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t) 1(t)$$

Second order step response – Underdamped and Undamped

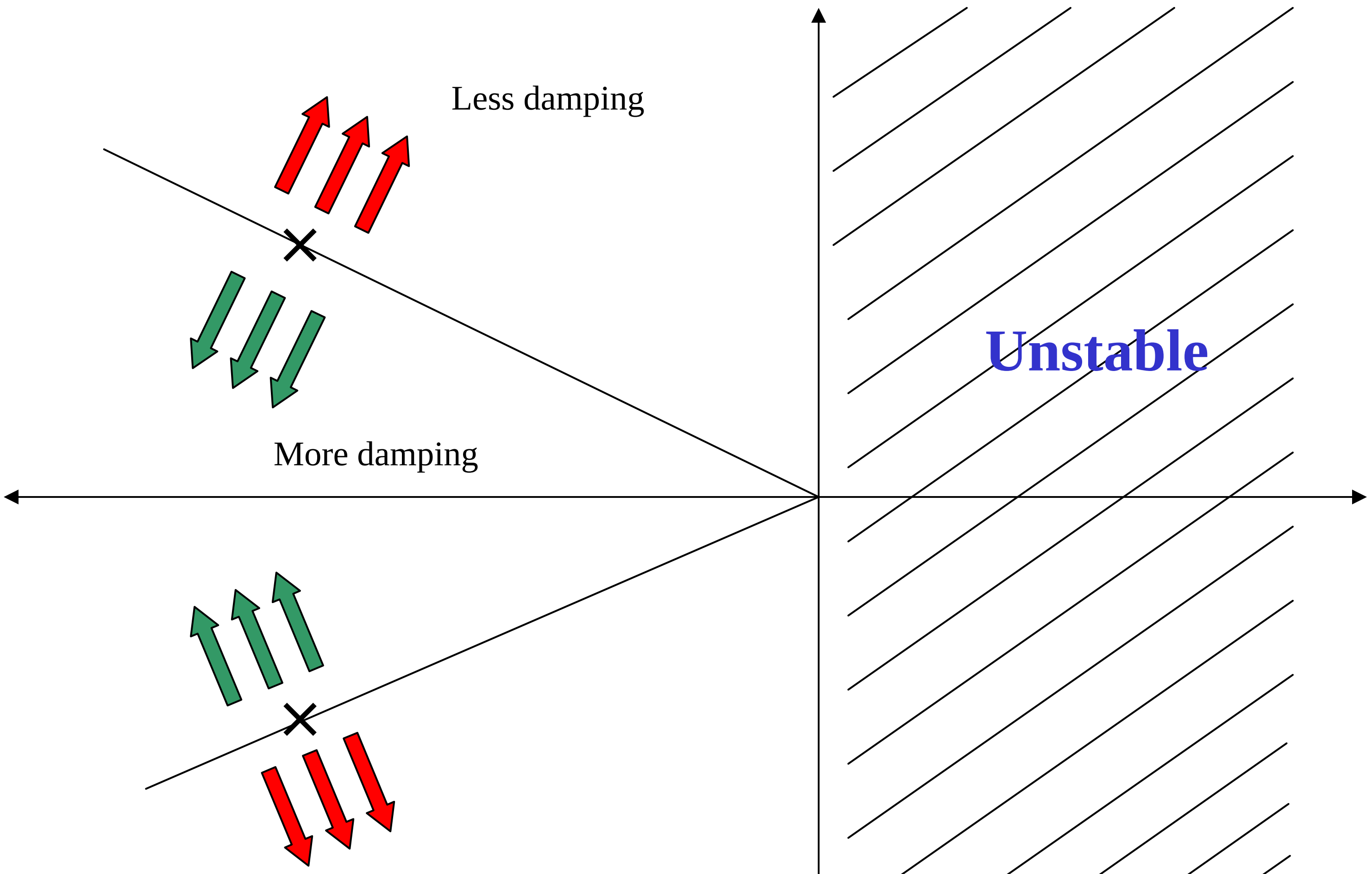


$$y_{step}(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\omega_n t + \theta)\right) 1(t)$$

Second order impulse response – Underdamped and Undamped

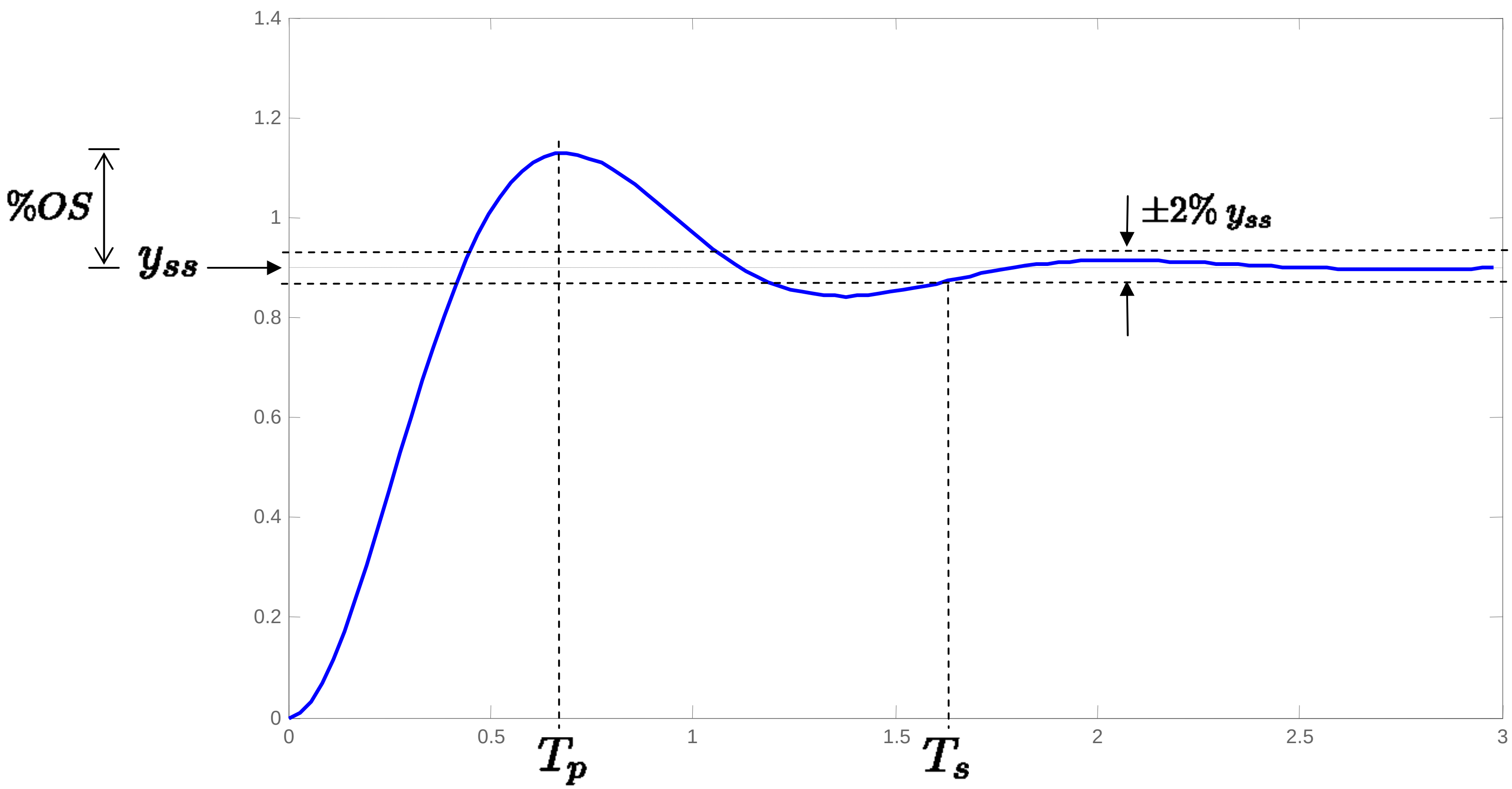


Second order impulse response – Underdamped and Undamped



Second order step response – Time specifications.

$$1(t) \longrightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\xi\omega_n\, s + \omega_n^2}} \longrightarrow y_{step}(t)$$



Second order step response – Time specifications.

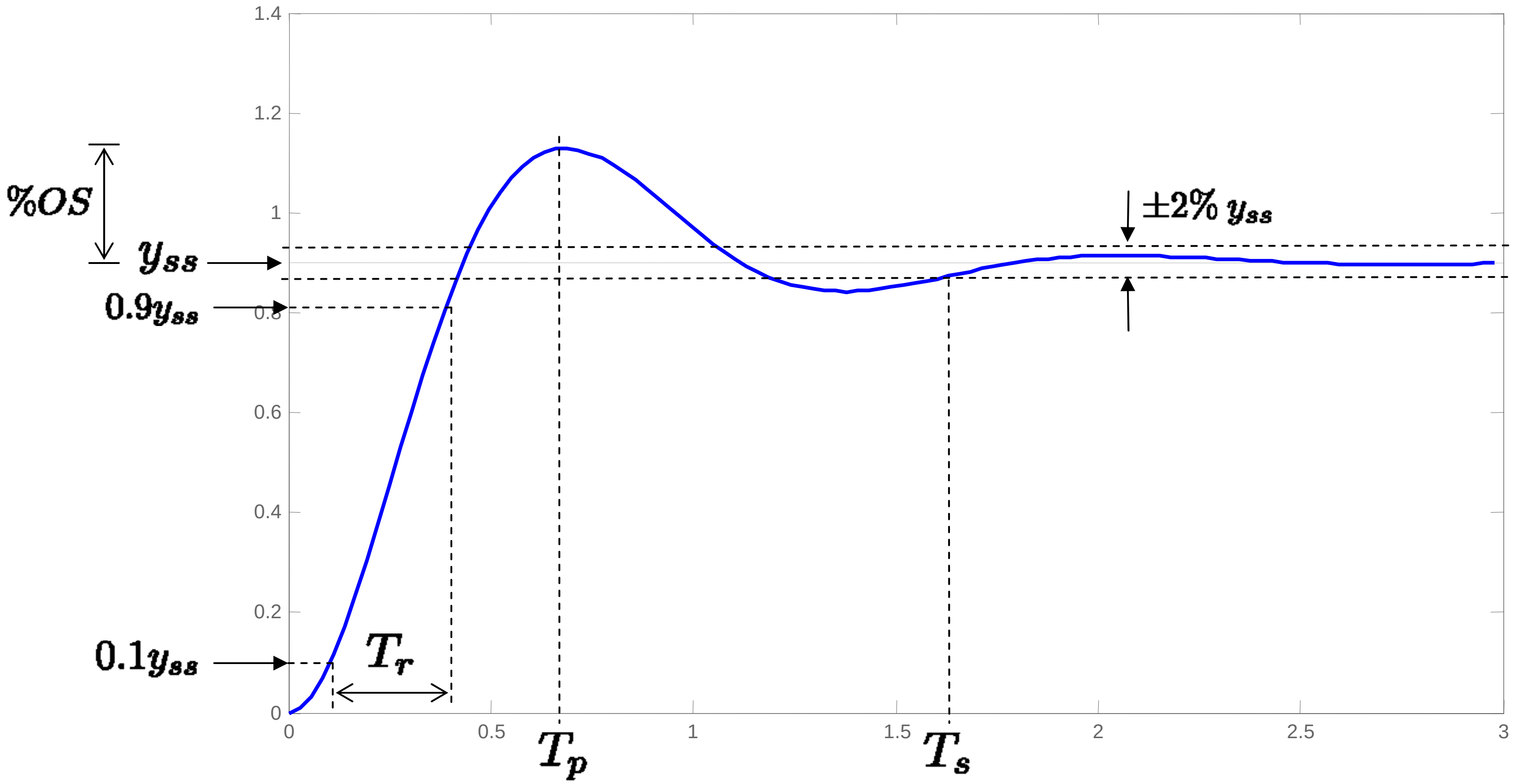
y_{ss} … Steady state value.

T_p … Time to reach first peak (undamped or underdamped only).

$\%OS$ … % of $y_{step}(T_p)$ in excess of y_{ss} .

T_s … Time to reach and stay within 2% of y_{ss} .

T_r … Time to rise from 10% to 90% of y_{ss} .



Second order step response – Time specifications.

- y_{ss} … **Steady state value**.

$$y_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\left(\frac{1}{s}\right)H(s) = \lim_{s\to 0} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2} = 1$$

More generally, if the numerator is not ω_n^2, but some K :

$$H(s) = \frac{K}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \Rightarrow \quad \boxed{y_{ss} = \frac{K}{\omega_n^2}}$$

Second order step response – Time specifications.

- T_p … **Peak time**.

$$
\begin{aligned}
\dot{y}_{step} &= \mathcal{L}^{-1}\Big[\, s\,Y(s) \Big] = \mathcal{L}^{-1}\Big[\, s\,\frac{1}{s}\,H(s) \Big] = \mathcal{L}^{-1}\Big[\, H(s) \Big] \\
&= \mathcal{L}^{-1}\Big[\, \frac{K}{s^2 + 2\xi\omega_n s + \omega_n^2} \Big] \\
&= \frac{K}{\omega_n\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t}\, \sin(\omega_n\sqrt{1-\xi^2}\, t)\, 1(t)
\end{aligned}
$$

Therefore,

$$
\dot{y}_{step} = 0 \quad \Leftrightarrow \quad \sin(\omega_n\sqrt{1-\xi^2}\, t) = 0
$$

$$
\Leftrightarrow \quad t = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}
$$

T_p is the time of the occurrence of the first peak $(n = 1)$:

$$
\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}
$$