Problems

- Poles and zeros

2-1. Find the poles and zeros of the following functions (including the ones at infinity, if any). Mark the finite poles with \( \times \) and the finite zeros with \( \circ \) in the s-plane.

(a) \( G(s) = \frac{10(s + 2)}{s^2(s + 1)(s + 10)} \)

(b) \( G(s) = \frac{10s(s + 1)}{(s + 2)(s^2 + 3s + 2)} \)

(c) \( G(s) = \frac{10(s + 2)}{s(s^2 + 2s + 2)} \)

(d) \( G(s) = \frac{e^{-2}}{10s(s + 1)(s + 2)} \)

- Laplace transforms

2-2. Find the Laplace transforms of the following functions. Use the theorems on Laplace transforms if applicable.

(a) \( g(t) = 5te^{-3u_c(t)} \)

(b) \( g(t) = (t \sin 2t + e^{-3t})u_c(t) \)

(c) \( g(t) = 2e^{-2t} \sin 2tu_c(t) \)

(d) \( g(t) = \sin 2t \cos 2t \)

(e) \( g(t) = \sum_{k=0}^{\infty} e^{-kt} \delta(t - kT) \) where \( \delta(t) = \text{unit-impulse function} \)

- Laplace transforms

2-3. Find the Laplace transforms of the functions shown in Fig. 2P-3. First, write a complete expression for \( g(t) \) and then take the Laplace transform. Let \( g_T(t) \) be the description of the function over the basic period and then delay \( g_T(t) \) appropriately to get \( g(t) \). Take the Laplace transform of \( g(T) \) to get \( G(s) \).

- Laplace transforms

2-4. Find the Laplace transform of the following function.

\[
g(t) = \begin{cases} 
0 & 0 \leq t < 1 \\
1 & 1 \leq t < 2 \\
2 - t & 2 \leq t < 3 \\
0 & t \geq 3
\end{cases}
\]

\[g(t)\]

\[g(t)\]

Figure 2P-3
2-5. Solve the following differential equations by means of the Laplace transform.

(a) \( \frac{d^2f(t)}{dt^2} + 5 \frac{df(t)}{dt} + 4f(t) = e^{-2t}u(t) \)  
Assume zero initial conditions.

(b) \( \frac{dx_1(t)}{dt} = x_2(t) \)
\( \frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) + u(t) \)
\( x_1(0) = 1, \ x_2(0) = 0 \)

2-6. Find the inverse Laplace transforms of the following functions. Perform partial-fraction expansion on \( G(s) \) first, then use the Laplace transform table. Use any computer program that is available for the partial fraction expansion.

(a) \( G(s) = \frac{1}{s(s + 2)(s + 3)} \)
(b) \( G(s) = \frac{10}{(s + 1)^2(s + 3)} \)
(c) \( G(s) = \frac{100(s + 2)}{s(s^2 + 4)(s + 1)} e^{-s} \)
(d) \( G(s) = \frac{2(s + 1)}{s(s^2 + s + 2)} \)
(e) \( G(s) = \frac{1}{(s + 1)^2} \)
(f) \( G(s) = \frac{2(s^2 + s + 1)}{s(s + 1.5)(s^2 + 5s + 5)} \)

2-7. Carry out the following matrix sums and differences. Do not use the computer, as these are straightforward.

(a) \( \begin{bmatrix} 5 & -6 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 10 & -3 \\ 2 & -3 \end{bmatrix} \)
(b) \( \begin{bmatrix} 2 & -2 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 3 & 4 \end{bmatrix} \)
(c) \( \begin{bmatrix} 1/s + 1 \\ 2/s \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -10 \end{bmatrix} \)

2-8. Determine if the following matrices are conformable for the products \( AB \) and \( BA \). Find the valid products.

(a) \( A = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \)
\( B = \begin{bmatrix} 6 & 1 & 0 \end{bmatrix} \)
(b) \( A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \)
\( B = \begin{bmatrix} 0 & 10 & -10 \\ -1 & 1 & 1 \end{bmatrix} \)
Note that the inverse Laplace transform of $\frac{1}{(s + p_1)^n}$ is given by

\[ \mathcal{L}^{-1}\left[\frac{1}{(s + p_1)^n}\right] = \frac{t^{n-1}}{(n-1)!} e^{-p_1 t} \]

The constants $a_{r+1}, a_{r+2}, \ldots, a_n$ in Equation (2-19) are determined from

\[ a_k = \left[ (s + p_k) B(s) \right] \frac{B(s)}{A(s)} \bigg|_{s = -p_k} \quad \text{for} \quad k = r + 1, r + 2, \ldots, n \]

The inverse Laplace transform of $F(s)$ is then obtained as follows:

\[ f(t) = \mathcal{L}^{-1}[F(s)] = \left[ b_1 + b_2 t + \cdots + \frac{b_{r-1}}{(r-2)!} t^{r-2} + \frac{b_r}{(r-1)!} t^{r-1} \right] e^{-p_1 t} \]

\[ + \left[ \frac{b_{r+1}}{(r+1)!} + \frac{b_{r+2}}{(r+2)!} t + \cdots + \frac{b_n}{n!} t^{n-1} \right] e^{-p_2 t} \quad \text{for} \quad t \geq 0 \]

**A-2-17.** Find the Laplace transform of the following differential equation:

\[ \ddot{x} + 3 \dot{x} + 6x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 3 \]

Taking the inverse Laplace transform of $X(s)$, obtain the time solution $x(t)$.

**Solution.** The Laplace transform of the differential equation is

\[ s^2 X(s) - sx(0) - \dot{x}(0) - 3sx(s) - 3x(0) + 6X(s) = 0 \]

Substituting the initial conditions and solving for $X(s)$,

\[ X(s) = \frac{3}{s^2 + 3s + 6} = \frac{2\sqrt{3}}{\sqrt{5}} \frac{\frac{\sqrt{15}}{2}}{(s + 1.5)^2 + \left(\frac{\sqrt{15}}{2}\right)^2} \]

The inverse Laplace transform of $X(s)$ is

\[ x(t) = \frac{2\sqrt{3}}{\sqrt{5}} e^{-1.5t} \sin\left(\frac{\sqrt{15}}{2} t\right) \]

**PROBLEMS**

**B-2-1.** Find the Laplace transforms of the following functions:

| (a) | \( f_1(t) = 0, \) for \( t < 0 \) | \( = 3 \sin (5t + 45^\circ) \) for \( t \geq 0 \) |
| (b) | \( f_2(t) = 0, \) for \( t < 0 \) | \( = 0.03(1 - \cos 2t) \) for \( t \geq 0 \) |

**B-2-3.** Obtain the Laplace transform of the function defined by

\[ f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t^2 e^{-t}, & \text{for } t \geq 0 \end{cases} \]
B-2-4. Obtain the Laplace transform of the function defined by

\[ f(t) = \begin{cases} 
0, & \text{for } t < 0 \\
\cos 2\omega t \cdot \cos 3\omega t, & \text{for } t \geq 0 
\end{cases} \]

B-2-5. What is the Laplace transform of the function \( f(t) \) shown in Figure 2–5?

B-2-6. Obtain the Laplace transform of the function \( f(t) \) shown in Figure 2–6.

\[ F(s) = \frac{10}{s(s + 1)} \]

Verify this result by taking the inverse Laplace transform of \( F(s) \) and letting \( t \to \infty \).

B-2-7. Find the Laplace transform of the function \( f(t) \) shown in Figure 2–7. Also, find the limiting value of \( \mathcal{L}[f(t)] \) as \( s \) approaches zero.

B-2-8. By applying the final-value theorem, find the final value of \( f(t) \) whose Laplace transform is given by

\[ F(s) = \frac{1}{(s + 2)^2} \]

determine the values of \( f(0+) \) and \( f(0+) \). (Use the initial-value theorem.)

B-2-9. Given

\[ F(s) = \frac{s + 1}{s(s^2 + s + 1)} \]

B-2-10. Find the inverse Laplace transform of

\[ F_1(s) = \frac{6s + 3}{s^2} \]

(b) \[ F_2(s) = \frac{5s + 2}{(s + 1)(s + 2)^2} \]

B-2-11. Find the inverse Laplace transforms of the following functions:

\[ F(s) = \frac{1}{s^2(s^2 + \omega^2)} \]

B-2-12. Find the inverse Laplace transform of

\[ 2\ddot{x} + 7\dot{x} + 3x = 0, \quad x(0) = 3, \quad \dot{x}(0) = 0 \]

B-2-13. What is the solution of the following differential equation?

\[ \ddot{x} + 2\dot{x} = \delta(t), \quad x(0-) = 0 \]

B-2-14. Solve the differential equation

\[ \ddot{x} + 2\dot{x} + \omega_n^2x = 0, \quad x(0) = a, \quad \dot{x}(0) = b \]

where \( a \) and \( b \) are constants.

B-2-15. Solve the following differential equation:

\[ \ddot{x} + 2\omega_n\dot{x} + \omega_n^2x = 0, \quad x(0) = a, \quad \dot{x}(0) = b \]

B-2-16. Obtain the solution of the differential equation

\[ \dot{x} + ax = A \sin \omega t, \quad x(0) = b \]