4.2 Determine the centroid of the line $y = x^2$ stretching from the origin to the point (2,4)m.

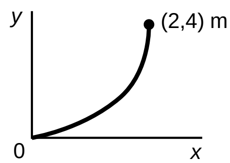

FIGURE P4.2

Solution: From figure S4.1 the differential element of length can be expressed as $dL = \sqrt{dx^2 + dy^2}$. The equation of the line is $y = x^2$ or upon differentiating: $dy = 2xdx$. Substitution of this into dL yields $dL = \sqrt{4x^2dx^2 + dx^2} = (\sqrt{1 + 4x^2})dx$. Now from the definition

$$x_C = \frac{1}{\int_l dL} \int_L x dL = \frac{\int_0^2 x\sqrt{1 + 4x^2}dx}{\int_0^2 \sqrt{1 + 4x^2}dx} = \underline{1.239 \text{ m}}$$

where the integral has been evaluated by Mathcad (see computer section) but may be evaluated by tables, or other programs (Mathematica, MATLAB, etc.). Likewise

$$y_c = \frac{1}{\int_L dL} \int_L y dL = \frac{\int_0^2 x^2\sqrt{1 + 4x^2}dx}{\int_0^2 \sqrt{1 + 4x^2}dv} = \underline{1.823 \text{ m}}$$

4.3 Repeat problem 4.2 for the case that the line extends to the point $x = 10$, $y = 100$ m.

Solution: Just change the upper limits of integration in the previous problem to 10. Using the Mathcad program requires no retyping only changes from 2 to 10:

$$x_c = 6.622 \text{ m} \quad y_c = 49.605 \text{ m}$$

4.4 Determine the centroid of the line $y = x^3$ stretching from the origin to the point (2,8)m.

Solution: Again the element of a differential length is $dL = \sqrt{dx^2 + dy^2}$. The relationship between y and x is $y = x^3$ so that $dy = 3x^2 dx$ and $dL = \sqrt{1 + 9x^4}dx$. Using this and $y = x^3$ in the definition of the centroids yield

$$x_c = \frac{\int_L x dL}{\int_L dL} = \frac{\int_0^2 x\sqrt{1+9x^4}dx}{\int_0^2 \sqrt{1+9x^4}dx} = \underline{1.426 \text{ m}}$$

$$y_c = \frac{\int_L y dL}{\int_L dL} = \frac{\int_0^2 x^3\sqrt{1+9x^4}dx}{\int_0^2 \sqrt{1+9x^4}dx} = \underline{3.744 \text{ m}}$$

where the integrals have been evaluated numerically (see the integral computational window).

4.5 Determine the centroid of the line $y = \sin x$ stretching from the origin to the point $(\pi/2, 1)$ mm. Make sure to label the units.

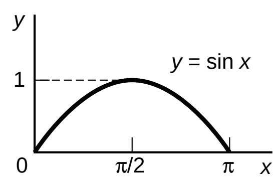

FIGURE P4.5

Solution: Again the differential element of the length of the line is $d\ell = \sqrt{dx^2 + dy^2}$. The relationship between x and y is $y = \sin x$ so that $dy = \cos x dx$ and $dy^2 = \cos^2 x dx$ so that $d\ell = \sqrt{1 + \cos^2 x}dx$. Substitution for $d\ell$ and y in terms of x into equations 4S3.5 yields

$$x_c = \frac{\int_0^{\pi/2} x\sqrt{1+\cos^2 x}dx}{\int_0^{\pi/2} \sqrt{1+\cos^2 x}dx} = 0.731 \text{ mm}$$

$$y_c = \frac{\int_0^{\pi/2} (\sin x)(\sqrt{1+\cos^2 x})dx}{\int_0^{\pi/2} \sqrt{1+\cos^2 x}dx} = 0.601 \text{ mm}$$

where the integrals are evaluated numerically using Mathcad. Please encourage students to see if the numbers make sense.

4.10 Calculate the centroid of the rectangular area shown relative to the coordinate system illustrated by direct integration.

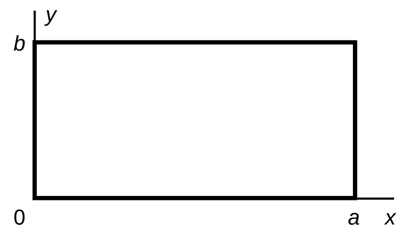

FIGURE P4.10

Solution:

$$x_c = \frac{\int_0^b \int_0^a x dx dy}{\int_0^b \int_0^a dx dy} = \frac{\int_0^b \frac{x^2}{2}\Big|_0^a dy}{ab} = \frac{\frac{a^2 b}{2}}{ab} = \underline{\frac{a}{2}}$$

$$y_c = \frac{\int_0^b \int_0^a y dx dy}{ab} = \frac{\int_0^a \frac{y^2}{2}\Big|_0^b dy}{ab} = \frac{\frac{b^2}{2}(a)}{ab} = \underline{\frac{b}{2}}$$

which again is the point of intersection of the two axis of symmetry.

4.11 Compute the centroid of the area of a triangle of height h and base b, using the coordinate system illustrated.

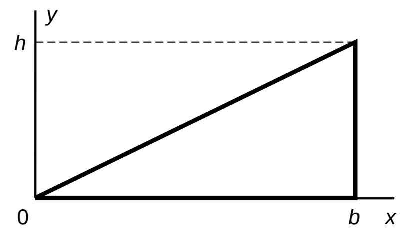

FIGURE P4.11

Solution: Using a single integral, write dA is at the area of a vertical strip between the two lines $y = hx/b$ and $y = 0$ where x ranges from 0 to b. Then following a typical calculus book approach the centroid for a plane area bounded between $y = f(x)$ above and $y = y(x)$ below is

$$x_c = \frac{\int x[F(x) - y(x)]dx}{\int [F(x) - y(x)]dx}$$

and

$$y_c = \frac{\frac{1}{2}\int [F^2(x) - y^2(x)]dx}{\int [F(x) - y(x)]dx}.$$

Thus

$$x_c = \frac{\int_0^b x[\frac{hx}{b} - 0]dx}{\int_0^b [\frac{hx}{b} - 0]dx} = \frac{\frac{h}{b}\frac{x^3}{3}|_0^b}{\frac{hx^2}{2b}|_0^b} = \frac{2}{3}\frac{b^3}{b^2} = \underline{\frac{2b}{3}}$$

$$y_c = \frac{\int_0^b \frac{1}{2}[(\frac{hx}{b})^2 - 0^2]dx}{\frac{1}{2}hb} = \frac{\frac{h^2 X^3}{3b^2}|_0^b}{hb} = \underline{\frac{1}{3}h}$$

4.18 Calculate the area and the centroid of the area between the two curves illustrated. The top curve is $y = \frac{h}{b}x$ and the bottom curve is $y = \frac{h}{b^2}x^2$, between the origin and the point (b, h).

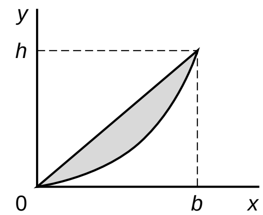

FIGURE P4.18

Solution: The element of area for a multiple integration is $dA = dydx$. Integrating first along y from $y = hx^2/b^2$ to $y = hx/b$, and then along x from 0 to b yields

$$A = \int dA = \int_0^b \int_{hx^2/b^2}^{hx/b} dydx = \underline{\frac{hb}{6}}$$

$$x_c = \frac{1}{A}\int_0^b \int_{hx^2/b^2}^{hx/b} xdydx = \underline{\frac{b}{2}}$$

$$y_c = \frac{1}{A}\int_0^b \int_{hx^2/b^2}^{hx/b} ydydx = \underline{\frac{2h}{5}}$$

where the integrals have been evaluated symbolically.

4.24 Calculate the volume and centroid for the parabolic "cone" illustrated. The equation of the line in the $x - y$ plane is $y^2 = \frac{R^2}{h}x$.

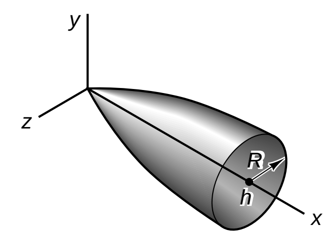

FIGURE P4.24

Solution: The volume element is, following the method of problem 4.25 $dV = \pi r^2 dx$ where $r = y$ so that $r^2 = y^2 = \frac{R^2}{h}x$ and $V = \int_0^h \pi \frac{R^2}{h} x dx = \underline{\frac{\pi R^2 h}{2}}$ The centroids along z and y are at zero. That along x is found from

$$\overline{x} = \frac{1}{V}\int_0^h (x)\pi \frac{R^2}{h} x dx = \frac{\pi R^2 h^2}{3} / \frac{\pi R^2 h}{2} = \underline{\frac{2}{3}h}$$

4.39 A mounting bracket is machined out of a flat piece of metal. This changes the centroid, and hence center of mass of the piece. Compute the new centroid and compare it to that of the same piece without the holes. The dimension in the figure are all in meters. The holes are all of radius 0.05 m.

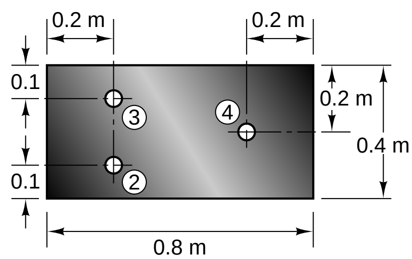

FIGURE P4.39

Solution: Denote the centroid and area of the bracket by the subscript 1 so that $x_1 = .2$ m, $y_1 = .4$ and $dA_1 = .32$. The holes are each of radius $r = .05$ m so that $x_2 = .2$, $y_2 = .1$ and $A_2 = \pi(.05)^2 = .0079$. Likewise $x_3 = .2$m, $y_3 = .3$m, $A_3 = -\pi(.05)^2$, $x_4 = .6$, $y_4 = .2$m and $A_4 = -.0079$ m^2

$$A = \sum_{i=1}^{4} A_i = \underline{0.296 \text{ m}^2},$$

$$x = \frac{1}{A}\sum_{i=1}^{4} x_i A_i = \underline{0.405 \text{ m}}$$

which is very close to its value (.4) before the holes are drilled, and

$$y = \frac{1}{A}\sum y_i A_i = \underline{0.2 \text{ m}}$$

which is exactly the same as before the holes are drilled. This is because of the symmetry that remains about this line $y = .2$ after drilling the holes.

4.42 Calculate the centroid of the cross section of the "I" beam.

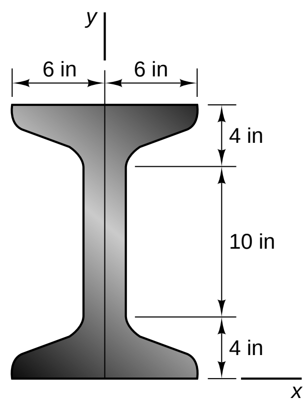

FIGURE P4.42

Solution: Because of symmetry, the centroid is at the intersection of the two axis of symmetry or $\underline{\overline{x} = 0''}$, $\underline{\overline{y} = 9''}$. A good problem to throw in to make sure the students can see the forrest from the trees.

4.43 A mounting bracket is manufactured out of 1/4-in. wide steel and formed into the shape illustrated. Compute its centroid. Does the centroid lie on the object?

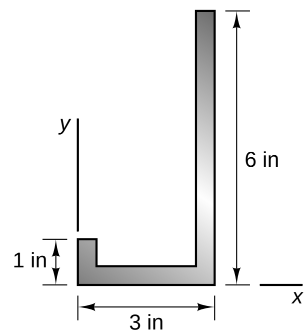

FIGURE P4.43

Solution: there are 3 rectangles to consider: $A_1 = 6 \times \frac{1}{4} = 1.5\text{in}^2$, $x_1 = 2\frac{7}{8}''2.875''$, $y_1 = 3''$, $A_2 = 3 \cdot \frac{1}{4} = .75''$, $x_2 = 1.5'', y_2 = \frac{1}{8} = .125'', A_3 = (1)(\frac{1}{4}) = .125\text{m}^2$, $x_3 = .125''$, $y_3 = .5$.

$$A = \sum_{i=1}^{3} A_i = \underline{2.375 \text{ in}^2},$$

$$x = \frac{1}{A} \sum_{i=1}^{3} x_i A_i = \underline{2.296 \text{ in}},$$

$$y = \frac{1}{A} \sum_{i=1}^{3} y_i A_i = \underline{1.961 \text{ in}}$$

which is not on the object itself.

Volumes

4.47 Two concrete columns are connected together to form a support structure for a dock. Compute the centroid of the composite body. The dimensions are in meters.

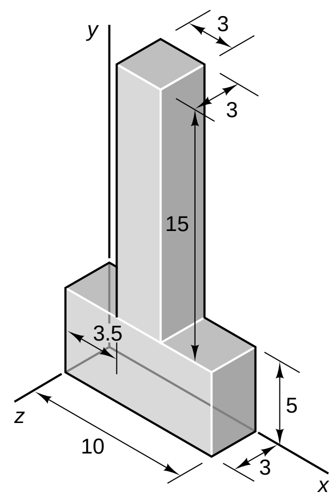

FIGURE P4.47

Solution: The volume of the horizontal column is

$$V_1 = (3)(5)(10) = 150 \text{ m}^3,$$

with centroid:

$$x_1 = \frac{10}{2} = 5\text{m},$$
$$y_1 = \frac{5}{2} = 2.5\text{m},$$
$$z_1 = \frac{3}{2} = 1.5 \text{ m}.$$

Likewise for the vertical column,

$$V_2 = (3)(3)(5) = 45\text{m}^3,$$

$x_2 = 3.5 + \frac{3}{2} = 5\text{m}$, $y_2 = 5 + \frac{15}{2} = 12.5$ m and $z_2 = \frac{3}{2} = 1.5$ m. Thus

$$V = \sum_{i=1}^{2} V_i = 150 + 45 = \underline{195 \text{ m}^3},$$

(continued)

$$\overline{x} = \frac{1}{V}(x_1V_1 + x_2V_2) = \frac{1}{195}((5)(150) + (5)(45)) = \underline{5\text{ m}},$$

$$y = \frac{1}{V}(y_1V_1 + y_2V_2) = \frac{1}{195}((2.5)(150) + (12.5)(45)) = \underline{4.808\text{ m}},$$

$$z = \frac{1}{V}(z_1V_1 + z_2V_2) = \frac{1}{195}((1.5)(150) + (1.5)(45)) = \underline{1.5\text{ m}}.$$

Note that the location of $\overline{x}$ and $\overline{z}$ are obvious from symmetry.