1. The Concept and Representation of Periodic Sampling of a CT Signal
2. Analysis of Sampling in the Frequency Domain
3. The Sampling Theorem — the Nyquist Rate
4. In the Time Domain: Interpolation
5. Undersampling and Aliasing
SAMPLING

We live in a continuous-time world: most of the signals we encounter are CT signals, e.g. $x(t)$. How do we convert them into DT signals $x[n]$?

— Sampling, taking snap shots of $x(t)$ every $T$ seconds.

$T$ – sampling period
$x[n] \equiv x(nT), n = ..., -1, 0, 1, 2, ...$ — regularly spaced samples

Applications and Examples
— Digital Processing of Signals
— Strobe
— Images in Newspapers
— Sampling Oscilloscope

How do we perform sampling?
Why/When Would a Set of Samples Be Adequate?

• Observation: *Lots* of signals have the same samples

• By sampling we throw out lots of information
  – all values of $x(t)$ between sampling points are lost.

**Key Question for Sampling:**
Under what conditions can we reconstruct the original CT signal $x(t)$ from its samples?
Impulse Sampling — Multiplying $x(t)$ by the sampling function

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$
Analysis of Sampling in the Frequency Domain

\[ x_p(t) = x(t) \cdot p(t) \]

Multiplication Property \( \Rightarrow X_p(j\omega) = \frac{1}{2\pi} X(j\omega) \ast P(j\omega) \)

\[ P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \]

\[ \omega_s = \frac{2\pi}{T} = \text{Sampling Frequency} \]

\[ \downarrow \]

\[ X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) \ast \delta(\omega - k\omega_s) \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \]
Illustration of sampling in the frequency-domain for a band-limited ($X(j\omega)=0$ for $|\omega| > \omega_M$) signal

$X_p(j\omega)$ drawn assuming

$\omega_s - \omega_M > \omega_M$

i.e. $\omega_s > 2\omega_M$

No overlap between shifted spectra
Reconstruction of $x(t)$ from sampled signals

If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$
The Sampling Theorem

Suppose \( x(t) \) is bandlimited, so that

\[
X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_M
\]

Then \( x(t) \) is uniquely determined by its samples \( \{x(nT)\} \) if

\[
\omega_s > 2\omega_M = \text{The Nyquist rate}
\]

where \( \omega_s = 2\pi/T \)
In practice, we obviously don’t sample with impulses or implement ideal lowpass filters.

— One practical example: The Zero-Order Hold
(2) Sampling is fundamentally a *time-varying* operation, since we multiply \( x(t) \) with a time-varying function \( p(t) \). However, 

\[
p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)
\]

is the identity system (which is \( TI \)) for bandlimited \( x(t) \) satisfying the sampling theorem \((\omega_s > 2\omega_M)\).

(3) What if \( \omega_s \leq 2\omega_M \)? Something different: more later.
Time-Domain Interpretation of Reconstruction of Sampled Signals — Band-Limited Interpolation

\[
p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT)
\]

\[
x(t) \overset{\times}{\longrightarrow} x_p(t) \overset{H(j\omega)}{\rightarrow} x_r(t)
\]

\[
x_r(t) = x_p(t) * h(t) \quad \text{where} \quad h(t) = \frac{T \sin \omega_c t}{\pi t}
\]

\[
= \left( \sum_{n=-\infty}^{\infty} x(nT) \delta(t-nT) \right) * h(t)
\]

\[
= \sum_{n=-\infty}^{\infty} x(nT) h(t-nT) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin[\omega_c(t-nT)]}{\pi(t-nT)}
\]

The lowpass filter interpolates the samples assuming \( x(t) \) contains no energy at frequencies \( \geq \omega_c \)
The LPF smooths out sharp edges and fills in the gaps.
Interpolation Methods

• Bandlimited Interpolation
• Zero-Order Hold
• First-Order Hold — Linear interpolation

\[ p(t) = \sum_{n = -\infty}^{+\infty} \delta(t - nT) \]
Demo: Sampled Images

Original Image

Sampled Image

ZeroOrder Hold

FirstOrder Hold

Anti aliased image

Anti aliased ZeroOrder Hold

Anti aliased First Order Hold
Undersampling and Aliasing

When $\omega_s \leq 2 \omega_M \Rightarrow$ Undersampling
Undersampling and Aliasing (continued)

Higher frequencies of $x(t)$ are “folded back” and take on the “aliases” of lower frequencies.

Note that at the sample times, $x_r(nT) = x(nT)$

$X_r(j\omega) \neq X(j\omega)$

Distortion because of aliasing

— Higher frequencies of $x(t)$ are “folded back” and take on the “aliases” of lower frequencies
— Note that at the sample times, $x_r(nT) = x(nT)$
A Simple Example

\[ x(t) = \cos(\omega_0 t + \phi) \]

\[ X(j\omega) = (\pi e^{-j\phi}) \quad (\pi e^{j\phi}) \]

\[ -\omega_0 \quad \omega_0 \]

\[ \omega_s = 3\omega_0 > 2\omega_M \]

\[ X_p(j\omega) \]

\[ -\omega_s \quad -\omega_0 \quad \omega_0 \quad \omega_s \quad \omega_s/2 \quad \omega_s - \omega_0 \]

\[ \omega \]

Aliased

\[ \omega_s = 1.2\omega_0 < 2\omega_M \]

Demo: Sampling and reconstruction of \( \cos \omega_0 t \)