Source-Aware Nonuniform Information Transmission for Minimum Distortion

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Abstract—This letter considers average input-output distortion minimization through joint optimization of source index assignment and modulation design. First, we derive the general optimization criterion and discuss simultaneous minimization of bit-error rate (BER) and distortion. Second, we propose a novel source-aware information transmission approach by exploiting the nonuniformity in Gray-coded constellations. The proposed approach makes it possible for simultaneous BER and distortion minimization and outperforms the existing schemes with big margin when channel coding is involved. Third, optimal source-aware constellation design for minimum distortion is proposed by incorporating the source information into constellation design. The power efficiency of the proposed source-aware nonuniform transmission scheme makes it attractive for any digital systems with analog inputs, particularly for systems with tight power constraints such as wireless sensor networks and space communications.

Index Terms—Constellation design, distortion minimization, nonuniform information transmission.

I. INTRODUCTION

In most modern communication systems, source representation is designed disjointly from information transmission. After A/D conversion of analog source signals, the bit streams are then uniformly encoded and mapped to symbols prior to transmission, uniform bit-error rate (BER) has been serving as one of the most commonly used performance measures. However, for systems with analog inputs, the ultimate goal of the communication system is to minimize the average input-output distortion. While BER plays the dominant role in distortion minimization, a communication system that minimizes the BER does not necessarily minimize the average input-output distortion. Therefore, novel system design that can achieve simultaneous BER and distortion minimization is highly desirable.

In [1], with no specifications on the modulation schemes, Zeger proposed a locally optimal index assignment (source coding) solution for minimum input-output distortion. On the other hand, assuming source index assignment has been done separately, and with the observation that the bits that come out of the source encoder are generally nonuniform (i.e., have different levels of significance), Masnick and Wolf [2] and Cover [3] introduced nonuniform modulation (index mapping) schemes, known as unequal error protection codes, for which the more important bits have lower error rate than other bits. Stemmed from [2] and [3], unequal error protection through both symmetric and asymmetric constellations have been further developed in [4]–[6]. It should be pointed out that the resulted constellation codeword design in [4]–[6] may no longer be Gray codes.

In this letter, taking a mixed analog-digital perspective, we consider average input-output distortion minimization through joint optimization of source index assignment and modulation design. First, we derive the general optimization criterion and discuss the possibility of simultaneous minimization of BER and distortion. Second, based on the fact that Gray code ensures minimum BER when the channel error probability is sufficiently small, we propose a novel source-aware information transmission approach by exploiting the nonuniformity in Gray-coded constellations. Third, optimal source-aware constellation design for minimum distortion is proposed by incorporating the source information into constellation design. The proposed source-aware nonuniform transmission scheme can be applied to any digital systems with analog inputs, particularly for systems with tight power constraints such as wireless sensor networks and space communications.

II. PROBLEM FORMULATION

Consider a digital communication system with analog input, as shown in Fig. 1. Let \( x_k \) be the discrete-time analog input vector resulted from uniform sampling of a continuous signal \( x(t) \). \( x_k \) is first fed into a quantizer \( Q \), which is a mapping from \( E^n \) to a finite set \( P \subset \mathbb{R}^n \), given by \( Q: \mathbb{R}^n \rightarrow P \), where \( P = \{ P_0, P_1, \cdots, P_{M-1} \} \) is the quantization codebook with \( P_i \in \mathbb{R}^n \) for \( 0 \leq i \leq M-1 \). We assume that the size of \( P \) is \( |P| = M = 2^m \), where \( m \geq 0 \) is an integer. Let \( y_k = Q(x_k) \) denote the quantization value of \( x_k \). \( y_k \) is coded into a binary sequence through an index assignment function \( \pi \), and is then fed into a source-aware digital channel encoder and a modulator, i.e., the most significant bits (MSB) and least significant bits (LSB) may be treated distinctly. Let \( \hat{y}_k \) denote the receiver output, which is an estimate of the quantization value \( y_k \); the averaged input-output distortion is then given by

\[
D_0 = E \{ d(x_k, \hat{y}_k) \} \tag{1}
\]

where \( d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a nonnegative function that measures the distance between two vectors in \( \mathbb{R}^n \).
Consider the widely used mean-square distortion function 
\[ d(x, y) = |x - y|^2. \]
In this case, the optimal quantizer satisfies the well-known nearest neighbor and centroid conditions [1]. The overall distortion \( D_0 \) can then be decomposed into two parts, namely, the distortion due to quantization noise and the distortion due to channel noise [7], denoted as \( n_q \) and \( n_c \), respectively. That is,
\[ x_k - \hat{y}_k = (x_k - y_k) + (y_k - \hat{y}_k). \]

When the quantizer satisfies the centroid condition, \( E\{n_q\} = 0 \). Noting that the quantization noise and the channel noise are independent, we have \( E\{n_q n_c^T\} = E\{n_q n_c^T\} = 0 \). It then follows that
\[ D_0 = E\{|n_q|^2\} + E\{|n_c|^2\} = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} E\{[x_k - y_k]^2\} + E\{[y_k - \hat{y}_k]^2\}. \]  

When the quantizer is optimal, the distortion due to quantization error is minimized. Minimization of \( D_0 \) is thus reduced to minimizing the distortion only due to the channel noise
\[ D = E\{d(y_k, \hat{y}_k)\}. \]

In the sequel, we will discuss joint source index assignment and constellation codeword design for minimum distortion, as well as nonuniform transmission based on Gray-coded constellations.

III. JOINT SOURCE INDEX ASSIGNMENT AND CONSTELLATION CODE DESIGN

Write \( y_k \) as \( y_k = \hat{y}_k + e_k \), where \( y_k, \hat{y}_k \in P = \{P_0, P_1, \ldots, P_{M-1}\} \), and \( e_k \) is the estimation error. For \( 0 \leq i \leq M - 1 \), define \( E_i = \{P_i - P_j, 0 \leq j \leq M - 1\} \); it then follows that
\[ D = E\{|y_k - \hat{y}_k|^2\} = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} E\{d[y_k, \hat{y}_k]\} = \sum_{i=0}^{M-1} E\{d(P_i, y_k)\} = \sum_{i=0}^{M-1} \sum_{e_k \in E_i} |e_k|^2 p(e_k). \]

Here \( p(x) \) denotes the probability that \( x \) occurs.

For efficient transmission, each quantizer output \( y_k \) is first coded to a binary sequence and then mapped to a symbol in a constellation \( \Omega \). When the signal-to-noise ratio (SNR) is reasonably high, as it is for most useful communication systems, each transmitted symbol is more likely to be mistaken for one of its neighbors than for far more distant symbols. Therefore, to minimize the distortion \( D \), the optimal index assignment and constellation codeword design should map the neighboring quantization vectors from the quantization codebook \( P \) to neighboring symbols in constellation \( \Omega \). More specifically, the optimal 1-1 mapping \( S : P \rightarrow \Omega \) should satisfy the following condition.

**Claim 1**: For systems utilizing scalar quantizer with codebook \( P \) and a symmetric (two-dimensional) rectangular or square constellation \( \Omega \) with \( |\Omega| = |P| = 2^m \), \( m > 1 \), there is no 1-1 mapping \( S : P \rightarrow \Omega \) that satisfies **C1**.

For the 4-bit scalar quantizer discussed above, instead of 16-QAM, consider the one-dimensional constellation 16-AM with Gray code. Clearly, it is easy to find a 1-1 mapping \( S : P \rightarrow \Omega \) that satisfies **C1** and minimizes the BER and the distortion \( D \) simultaneously.
groups that differ by only one position. Here, we revisit the nonuniformity in constellations with Gray codes and introduce a nonuniform transmission scheme based on Gray-coded constellations. In the following, we illustrate the idea through Gray coded 16-QAM constellation shown in Fig. 3(b).

In 16-QAM, each codeword has the form \( k_3 b_2 b_1 b_0 \). If we go through the 16 symbols in Fig. 3(b), there are altogether 24 nearest neighbor bit changes, among which \( b_0 \) and \( b_2 \) each changes four times, and \( b_1 \) and \( b_2 \) each changes eight times. Letting \( P_e \) denote the error probability, then this implies that when SNR is reasonably high, \( P_e(b_0) = P_e(b_2) = (1/2)P_e(b_1) = (1/2)^2 P_e(b_2) \). Our analysis coincides with the theoretical results in [9]–[11].

Here we propose to minimize the average distortion by exploiting the inherent nonuniformity in Gray-coded constellations, i.e., to map the more significant bits from the source encoder to bit locations with lower error probability in constellations with Gray codes. For example, considering a 4-bit quantizer and a 16-QAM constellation as in Fig. 3(b), the two MSBs will be mapped to \( b_0 \) and \( b_2 \), while the two LSBs will be mapped to \( b_1 \) and \( b_3 \). The proposed approach can be applied to both symmetric and asymmetric constellations. To illustrate the performance, we compare the proposed Gray-code-based nonuniform transmission scheme with the block partitioning-based approaches [5], [6] for both coded and uncoded systems (note that the MR scheme [4] is only for asymmetric constellations and coincides with the block partitioning-based method in the symmetric case [5], [6]).

**Example 1:** The source is assumed to be analog with the amplitude uniformly distributed within \([0,100]\), quantized using a 12-bit uniform quantizer. First, each 12-bit quantization output \( k_3 b_2 \cdots b_1 \) (with \( b_0 \) being the MSB and \( b_3 \) being the LSB) is partitioned into three 4-bit strings: \( k_3 b_2 b_1 b_0, b_2 b_1 b_0, b_1 b_0 \) (with \( b_0 \) being the MSB and \( b_3 \) being the LSB), then mapped to both symmetric and asymmetric 16-QAM constellations based on the block partitioning (BP) scheme or the proposed Gray-code based nonuniform transmission scheme. By random index assignment, we mean that no distinction is made on MSBs and LSBs, and the strings are mapped to the Gray coded constellation based on their original bit arrangements \( k_3 b_2 b_1 b_0, b_2 b_1 b_0, b_3 b_1 b_0, b_1 b_0, b_2 b_1 b_0 b_1 \). The result is shown in Fig. 4(a).

**Example 2:** In this example, the impact of channel coding is investigated for both systematic and nonsystematic coding schemes. Using the same source as in Example 1, a 10-bit uniform quantizer is connected with a source-aware channel encoder, for which the first four MSBs are fed to a rate 1/3 convolutional (or Turbo) encoder, and the remaining 6 bits are fed to a rate 1/2 convolutional (or Turbo) encoder, respectively. The channel coding outputs are then mapped to 16-QAM constellations nonuniformly based on the block partitioning approach and the proposed mapping scheme, respectively. As shown in Fig. 4(b), the Gray-code-based nonuniform transmission outperforms the non-Gray-coded methods with big margins when channel coding is involved. The underlying arguments are: 1) channel coding may change the geometric structure of the unencoded symbols; 2) when SNR is reasonably high, BER of the more significant bits vanishes, and BER of the less significant bits dominates the overall distortion.

**B. Source-Aware Constellation Design**

Following our discussions in Sections III and IV-A, we propose to incorporate the source information reflected in optimal
Consider a nonuniform scalar quantizer with four possible quantization values. Assuming the quantization code book is $\hat{P} = \{P_1, \ldots, P_4\}$, where $P_i < P_{i+1}$ for $i = 1, 2, 3$ and each $P_i$ occurs with probability $p(P_i) = p_i$ for $i = 1, \ldots, 4$. Define $D_{ij} = \|P_i - P_j\|^2$. Along the lines of Claim 1, we consider the design of a 4-AM constellation $\Omega = \{A_1, \ldots, A_4\}$ and assume that the 1-1 mapping $S : \Omega \rightarrow P$ is designed to satisfy condition C1. We further assume that the quantizer is optimal, i.e., it satisfies the nearest-neighbor rule and the centroid criterion. Define $d_{ij} = \|A_i - A_j\|$ for $i = 1, 2, 3$. For $i = 1, \ldots, 4$, let $D_i = \sum_{j=1}^{4} A_{i(j)}A_{i(j)} p(A_i)$ be the average distortion corresponding to symbol $A_i$. The overall average distortion can be written as $D = \sum_{i=1}^{4} p_i D_i$. Define $\gamma_1 = d_1/\sigma$, $\gamma_2 = d_2/d_1$, $\gamma_3 = d_3/d_1$, the problem of optimal constellation design for minimum average distortion is reduced to finding $\gamma_1$, $\gamma_2$, $\gamma_3$ such that $D$ is minimized, subjected to a power constraint, i.e.,

$$\min_{\gamma_1, \gamma_2, \gamma_3} D_i \quad \text{subjected to} \quad P_s = C \quad (8)$$

where $P_s$ is the average symbol power, and $C$ is a constant. This method can readily be extended to more general cases. The proposed joint quantizer-constellation design scheme generalizes the concept of nonuniform constellation design from the perspective of joint source-channel coding.

V. CONCLUSIONS

This letter studied joint optimization of source index assignment and modulation design for overall input-output distortion minimizations in communication systems. The proposed scheme can be applied for higher power efficiency in any digital systems with analog inputs.

REFERENCES