

A Direct Method for Determination of Failure Frequency Indices Using State Space Decomposition

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Abstract: Of the many analytical methods used in the determination of steady state reliability indices of discrete capacity systems, the method of state space decomposition is one of the most effective, because of its ability to work with sets of states rather than individual states. In the past, decomposition-based methods have been used mostly for determining only probability indices; in those cases where they have been used to determine frequency and duration indices, the process has been cumbersome. This paper describes a technique for the determination of frequency and duration indices using the method of state space decomposition. For the first time direct, closed form expressions are developed and presented, and an efficient implementation is outlined. The approach is illustrated by means of a numerical example and validated using an enumeration-based method.

Key-Words: reliability indices, failure indices, frequency indices, duration indices, state space decomposition.

1 Introduction

There has been considerable research on the problem of reliability analysis of discrete capacity systems. The existing body of knowledge includes analytical methods, Monte Carlo simulation methods, and hybrid methods. In all of these methods, the underlying approach consists of first modeling the system behavior as a stochastic process, and then quantifying the system reliability in terms of the probability or frequency of encountering the failure states, or the period of time the system spends in these states. We thus have probability, frequency and duration indices to describe the reliability of a given system. These three indices are related in that the failure probability equals the product of the failure frequency and the failure duration. Of the three indices, probability indices are the most easily determined, but frequency and duration (F&D) indices convey a clearer and more complete understanding of the system behavior in terms of its reliability. For example, a system may on the average fail once a year and take an average of 88 hours to repair, while another system may fail once in 10 years and take 877 hours to repair; both systems, however, have failure probabilities of 0.01.

There are several analytical techniques for determining steady state reliability indices of systems. All of these techniques are based on identification (by enumeration or classification) of the failure states of the system. While in small systems it is often possible to exhaustively enumerate the failure states, large systems usually have such large numbers of states as to make enumeration impossible. In such cases it

becomes expedient to deal with sets of states rather than individual states, or to abandon the analytical approach altogether and use Monte Carlo simulation. Some hybrid methods use combinations of analytical and simulation techniques.

The method of state space decomposition is an analytical technique which classifies the set of all system states into disjoint sets of failure states and acceptable states. It is a recursive method, first developed by Doulliez and Jamoulle [1]¹, and applied by several others, such as in [2, 3, 4]. The method is generally applicable to most systems that can be modeled as flow networks. This includes pipeline networks, transportation networks, communication networks, computer networks, mechanical systems, and electrical networks such as power systems and electronic circuits. Most of these networks are discrete capacity systems; the rest are often modeled as discrete capacity systems to facilitate analysis.

In the past, the method of decomposition has been applied to discrete capacity systems, mostly for determination of probability indices. There are relatively few instances in the literature where F&D indices have been determined from methods dealing with sets of states, because it has been a difficult problem, and prior attempts [5, 6, 7] have used recursive techniques that tend to be cumbersome. In this paper we develop, for the first time, direct closed form expressions that simplify the determination of F&D indices to the point where it requires little additional computing effort beyond the determination of the probability indices.

¹RAIRO articles are in English or French; [1] is in English.

In the following sections we develop the theory, illustrate the technique by applying it to a sample system, and validate the results using the method of enumeration.

Notations:

A -state	functional (acceptable) state
L -state	loss of functionality (failed) state
A -sets	disjoint sets of A -states
L -sets	disjoint sets of L -states
N_L	number of L -sets
N_C	number of components in system
$P\{\cdot\}$	probability of set $\{\cdot\}$
$F\{\cdot\}$	frequency of set $\{\cdot\}$
λ, μ	transition rates
F^+	frequency of transition to higher capacity states
F^-	frequency of transition to lower capacity states
p_{kj}	probability of state k of component j
P_{kj}	cumulative probability of state k of component j
F_{kj}	cumulative frequency of state k of component j

2 Theory

We will proceed to develop the theory in three stages. First we will revisit the relevant features of the method of state space decomposition without actually describing the mechanics. Next, we will show how to determine the frequencies of encountering sets of states, and extend this to unions of sets. Finally, we will describe a means of setting up the data in such a manner as to simplify the computation of the indices.

2.1 State Space Representation

The state of a system is described by the states of all the individual components that constitute the system. If each of n components can assume several capacity levels, then a n -tuple of the prevailing capacity levels of the components would describe the prevailing state of the system. For a system of N_C components, if one were to define an N_C -dimensional space, each axis representing the different capacities a corresponding component can assume, then each point in this space would uniquely describe a state of the system. The space would constitute the complete set of states the system can assume, and would be known as the state space of the system. Such an N_C -dimensional Cartesian coordinate representation has been found to be convenient, and is in widespread use.

This representation can be illustrated using the following system. Consider a capacity constrained network, as shown in Fig. 1. The capacity of each sending arc is represented by the state transition diagram shown in Fig. 2. Each transmission arc (shown by a broken line) has a capacity of 100 in each direction and each receiving arc has a capacity of 100. Assume that $\lambda = 0.2$ failures/h, $\mu = 2$ repairs/h. Table 1 gives the cumulative probabilities and frequencies for this state transition diagram.

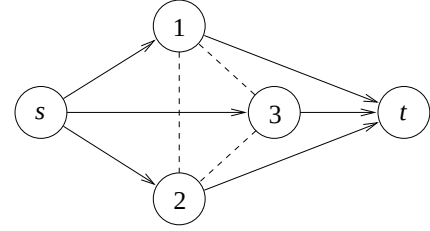


Figure 1: Three node network.

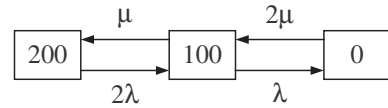


Figure 2: State transition diagram for each sending arc.

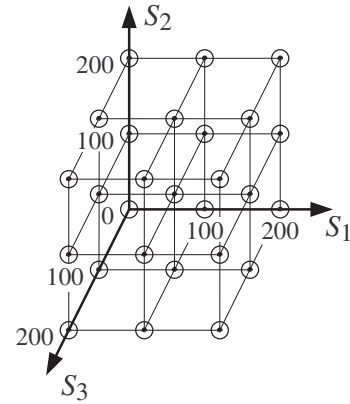


Figure 3: State space for three-component system.

Since there are three components with variable capacity, the system described has $3^3 = 27$ states. Fig. 3 shows the Cartesian representation of this state space.

In general, this form of representation is valid for both continuous and discrete capacity systems. Engineering systems are modeled as discrete capacity systems, and the methodology presented in this paper will analyze discrete capacity systems.

For most systems of realistic size, due to the multitude of states (e.g., the state space of a system of n components with m states each consists of m^n states),

Table 1: Probability and frequency distribution for each sending arc.

CAPACITY	PROBABILITY	CUMULATIVE PROBABILITY	CUMULATIVE FREQUENCY
200	8.2645×10^{-1}	1	0/h
100	1.6529×10^{-1}	1.7355×10^{-1}	$3.3058 \times 10^{-1}/h$
0	8.2645×10^{-3}	8.2645×10^{-3}	$3.3058 \times 10^{-2}/h$

it is impractical to store information (capacity, probability, frequency) about individual states separately; in such cases, states are dealt with in groups or sets. Sets of states are represented as hypercuboids in the N_C -dimensional space, and are denoted, for simplicity, by boundary vectors. (This will become clearer in the sections that follow.) However, sets of states that have certain common characteristics (such as failure states) do not necessarily conform to this shape, and must be represented as unions of several adjacent hypercuboids.

2.2 State Space Decomposition

The method of decomposition consists of classifying all points in the state space into disjoint sets of acceptable (A), unclassified (U) and failure (L) states. Each of these sets can be defined by a maximum state and a minimum state such that all states between and including these two constitute the given set. Each U -set is further decomposed into an A -set, U -sets and L -sets. This is continued until only A -sets and L -sets remain. The reliability indices are then determined from the L -sets. References [1–4] provide detailed descriptions of the decomposition method. A very brief description is provided here.

For the state space shown in Fig. 3, all the states are initially included in the first U -set, represented as $U_0 = \left\langle \begin{array}{ccc} 200 & 200 & 200 \\ 0 & 0 & 0 \end{array} \right\rangle$. All sets decomposed from U_0 will be denoted in this manner, where the bottom row provides the coordinates of the lower limit of the set and the top row denotes the upper limit.

The classification of states is achieved by identifying boundary vectors that partition the set being decomposed. The method of identification of boundary vectors is determined by the network model, and is specific to the kind of system or process being analyzed. For the flow network described in the illustration in section 2.1, the following boundary vectors may be defined.

1. The lower bound of acceptability, u :

Suppose all the capacities are set equal to the upper boundaries of the U -set and based on these constraints the system load curtailments are minimized; then the combination of the lowest capacity states which yield zero system curtailment

constitutes the u -vector. In other words, all states between and including the u -vector and the upper boundary of the U -set constitute an A -set.

2. The vector of minimum responsibility, v :

Suppose all the capacity states, except the i -th, are set equal to the corresponding elements of the upper boundary of the U -set; then the lowest value of the i -th member which yields zero system curtailment is the i -th component of the v -vector. The v -vector therefore comprises the minimum responsibilities of all the members.

These vectors are determined by solving suitably formed [1–4] maximal flow problems. For the illustrative case used in this paper, they can be manually determined with little effort. The u -vector is generally non-unique, but any valid solution will provide correct decomposition and eventually result in correct identification of L -sets. The complete decomposition of the state space for the illustrative case is shown in Fig. 4, with all the boundary vectors and resulting A , U and L sets. For this case the method completes in three recursions, shown as stages 1 through 3 in Fig. 4. The results are shown graphically in Fig. 5.

Now except in very rare cases, the bounding surface of the total set of failure states is very segmented. For instance, the L -sets in the illustrative three dimensional state space are as shown in Fig. 5. The total L -set is given by

$$L = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \cup L_6 \quad (1)$$

The subsets $L_1, \dots, L_6$ are obtained in the form of cuboids (hypercuboids in a multidimensional space) because the boundary vectors identify corner points of the sets. Another characteristic of these sets is that all the sets shown are disjoint.

The reliability indices are determined from the union of these sets. This is explained in the next section. It is now appropriate to state the conditions under which the state space can be subjected to decomposition. The state space is required to be coherent, i.e., the failure or degradation of a component cannot improve system performance, and, likewise, the improvement or restoration of a component cannot deteriorate system performance. If this condition is satisfied the following analysis may be applied.

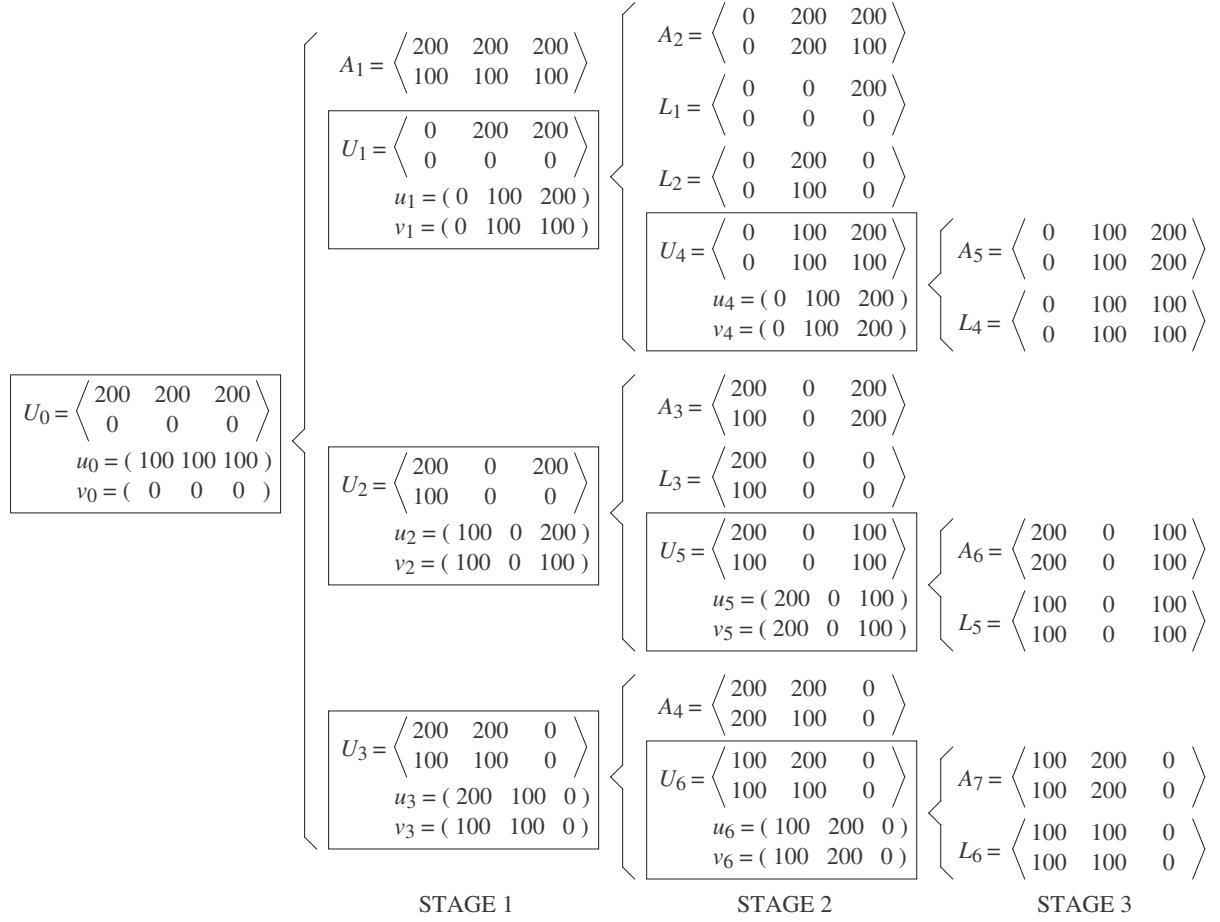


Figure 4: Stages of recursive decomposition.

2.3 Determination of Indices

For the purpose of the forthcoming analysis, we will use the following assumptions:

- The system is frequency-balanced [8], i.e., the frequency between any pair of states is equal in both directions. This automatically holds true if every component can be represented by a two-state Markov model. For multi-state components we assume that forced frequency balance is imposed as described by [8]. In the next section we will describe a method that automatically ensures frequency balance.
- The system can transit in one step from one state to another only by failure or restoration of a single component.

A transition that results from the repair of a failed component is said to be an upward transition; similarly, a transition that results from the failure of a functional component is said to be a downward transition.

Assume that decomposition has resulted in N_L failure sets; then the aggregate failure set is given by

$$L = \bigcup_{i=1}^{N_L} L_i \quad (2)$$

Clearly, since the sets are all disjoint, the failure probability is

$$P\{L\} = \sum_{i=1}^{N_L} P\{L_i\} \quad (3)$$

Now we develop a closed form expression for the failure frequency. Consider a state x in the set L_i . The sum of frequencies of upward and downward transitions from this state are given by (4) and (5), respectively:

$$F^+\{x\} = P\{x\} \sum_{j=1}^{N_C} \lambda_{xj}^+ \quad (4)$$

$$F^-\{x\} = P\{x\} \sum_{j=1}^{N_C} \lambda_{xj}^- \quad (5)$$

where λ_{xj}^+ : transition rate of component j from its state in system state x to higher capacity states;

λ_{xj}^- : transition rate of component j from its state in system state x to lower capacity states.

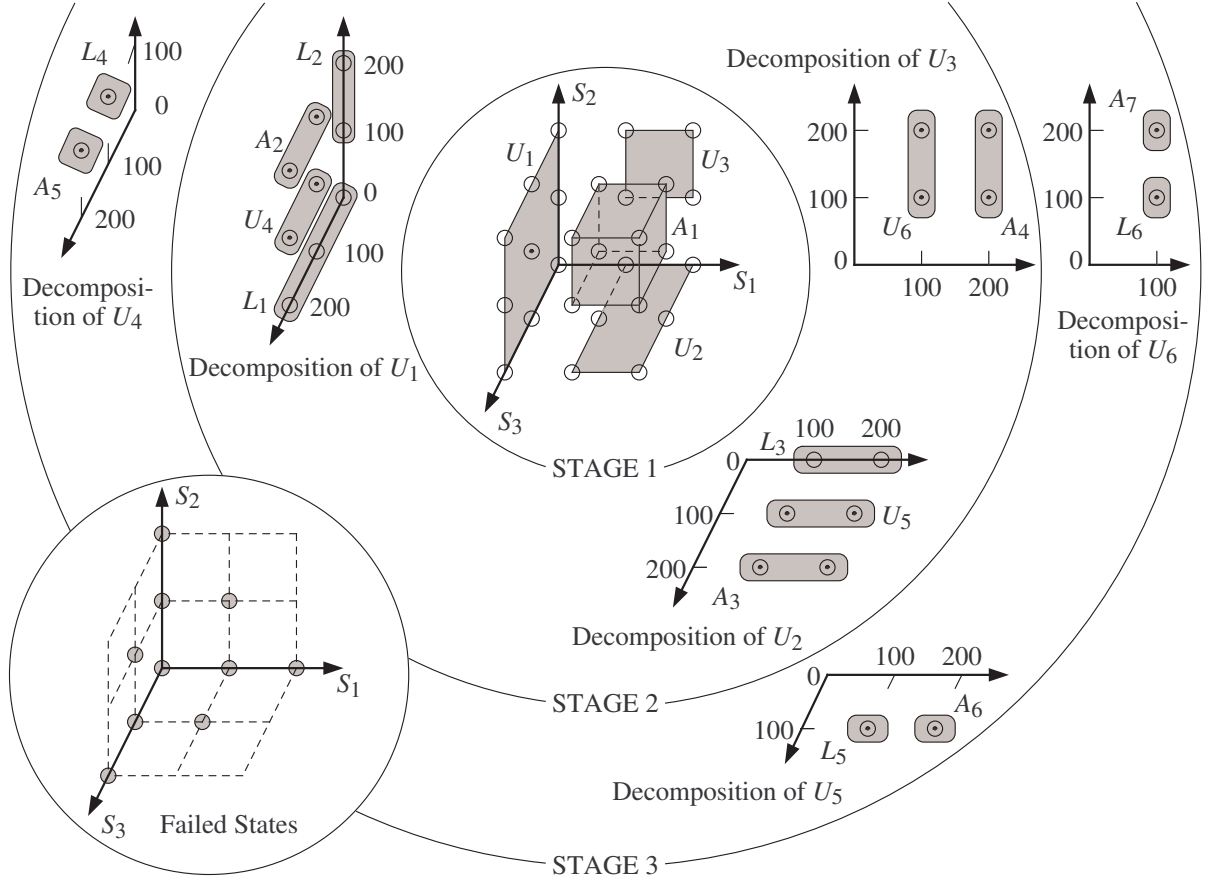


Figure 5: Recursive decomposition of state space.

Since the system is considered coherent, the $F^- \{x\}$ transitions cannot cross the boundary between L -sets and A -sets. The upward transitions can be split into two components,

$$F^+ \{x\} = F_0^+ \{x\} + F_1^+ \{x\} \quad (6)$$

such that the $F_0^+ \{x\}$ component is the one that crosses the boundary between the L -sets and A -sets and $F_1^+ \{x\}$ remains within the L -set. Clearly, the sum of the frequencies $F_0^+ \{x\}$ for all states x in all the L -sets would yield the frequency of the expected transitions across the boundary of L -sets and A -sets. It is known that when the state space is divided into two disjoint sets, the steady state frequency is equal in both directions [9, p. 74]. Therefore the frequency from L -sets to A -sets is the same as from A -sets to L -sets. Hence, the failure frequency is given by

$$F \{L\} = \sum_{i=1}^{N_L} \sum_{x \in L_i} [F_0^+ \{x\}] \quad (7)$$

In the next step we will develop (7) into a form that will obviate the explicit identification of transitions that cross the boundary.

Since the components of (6) are assumed independent and are frequency balanced for any two states x_i and x_j [8],

$$F \{x_i \rightarrow x_j\} = F \{x_j \rightarrow x_i\} \quad (8)$$

Because of (8), we have

$$\sum_{i=1}^{N_L} \sum_{x \in L_i} F_1^+ \{x\} = \sum_{i=1}^{N_L} \sum_{x \in L_i} F^- \{x\} \quad (9)$$

This can be understood as follows. Starting from a failed state, an upward transition that does not cross the boundary is actually an upward transition from a failed state to another failed state. For every upward transition from a failed state x_i to another failed state x_j , there is a downward transition from x_j to x_i , of equal frequency. Hence the two sums in (9) are equal. This also implies that

$$\sum_{i=1}^{N_L} \sum_{x \in L_i} [F_1^+ \{x\} - F^- \{x\}] = 0 \quad (10)$$

Combining (7) and (10), we get

$$F \{L\} = \sum_{i=1}^{N_L} \sum_{x \in L_i} [F_0^+ \{x\} + F_1^+ \{x\} - F^- \{x\}] \quad (11)$$

which, because of (6), can be rewritten as

$$F\{L\} = \sum_{i=1}^{N_L} \sum_{x \in L_i} [F^+\{x\} - F^-\{x\}] \quad (12)$$

or simply as

$$F\{L\} = \sum_{i=1}^{N_L} [F^+\{L_i\} - F^-\{L_i\}] \quad (13)$$

The components of (13) are determined from (14) and (15) below:

$$F^+\{L_i\} = \sum_{x \in L_i} \left[P\{x\} \sum_{j=1}^{N_C} \lambda_{xj}^+ \right] \quad (14)$$

$$F^-\{L_i\} = \sum_{x \in L_i} \left[P\{x\} \sum_{j=1}^{N_C} \lambda_{xj}^- \right] \quad (15)$$

Equation (13) is equivalent to equation (7) and is suitable for calculating the frequency of failure of the system, as shown in the next section. As mentioned before, when using (13), it is not necessary to explicitly identify transitions that cross the boundary between L -sets and A -sets.

Having obtained the probability and frequency of system failure, the determination of $T\{L\}$, the expected duration of system failure, is trivial:

$$T\{L\} = \frac{P\{L\}}{F\{L\}} \quad (16)$$

In the rest of this paper, therefore, we will only deal with failure probability and frequency.

2.4 Implementation Strategy

The equations (3) and (13) are used to determine the probability and frequency of system failure. We will now show how the data can be constructed so that calculation of $P\{L_i\}$ and $F\{L_i\}$ can be achieved with very little effort. This can be done as follows.

Consider a network with N_C components where each component has several capacity states, of which every state has a known probability and frequency of occurrence. Construct cumulative probability and frequency distributions for each component, so that P_{kj} and F_{kj} denote, respectively, the cumulative probability and frequency of state k of component j . The states are so indexed that a higher index implies a higher capacity. For this system, a state space is constructed, and decomposed into disjoint A -sets and L -sets, as exemplified in section 2.2.

Now consider a failure set L_i . Recall that L_i is described by a maximum state and a minimum state.

Assume that states n and m of component j define the maximum and minimum states of set L_i . For state k of component j , where $m \leq k \leq n$, the cumulative probability and frequency can be expressed as follows [8, 6].

$$P_{kj} = P_{(k-1)j} + p_{kj} \quad (17)$$

$$F_{kj} = F_{(k-1)j} + p_{kj} (\lambda_{kj}^+ - \lambda_{kj}^-) \quad (18)$$

where p_{kj} : probability of state k of component j ;
 $\lambda_{kj}^+, \lambda_{kj}^-$: transition rates from state k of component j to higher and lower capacity states respectively.

From (17) and (18), we get

$$\lambda_{kj}^+ - \lambda_{kj}^- = \frac{(F_{kj} - F_{(k-1)j})}{(P_{kj} - P_{(k-1)j})} \quad (19)$$

If we now represent the states from m to n of component j by an equivalent state s , then the equivalent values for this can be expressed in the same manner as (19):

$$\lambda_{sj}^+ - \lambda_{sj}^- = \frac{(F_{nj} - F_{(m-1)j})}{(P_{nj} - P_{(m-1)j})} \quad (20)$$

Similarly, the probability of this equivalent state can be written as

$$p_{sj} = P_{nj} - P_{(m-1)j} \quad (21)$$

In a similar fashion, for every component, the states between the maximum and minimum of set L_i can be represented by an equivalent state.

Then the probability of L_i is:

$$P\{L_i\} = \prod_{j=1}^{N_C} p_{sj} \quad (22)$$

and the frequency is:

$$F^+\{L_i\} - F^-\{L_i\} = P\{L_i\} \sum_{j=1}^{N_C} (\lambda_{sj}^+ - \lambda_{sj}^-) \quad (23)$$

Hence the probability of system failure is determined by substituting $P\{L_i\}$ from (22) into (3), and the frequency is determined by substituting $F^+\{L_i\} - F^-\{L_i\}$ from (23) into (13), as shown below.

$$F\{L\} = \sum_{i=1}^{N_L} P\{L_i\} \sum_{j=1}^{N_C} (\lambda_{sj}^+ - \lambda_{sj}^-) \quad (24)$$

Notice that the determination of the failure frequency using the expression given in (24) adds little to the computational effort expended in decomposing the state space and calculating the failure probability using (22).

This technique will now be illustrated using the example described in sections 2.1 and 2.2.

Table 2: Determination of the frequency of system failure.

	L_1	L_2	L_3	L_4	L_5	L_6
ΔF_1	3.3058×10^{-2}	3.3058×10^{-2}	-3.3058×10^{-2}	3.3058×10^{-2}	2.9752×10^{-1}	2.9752×10^{-1}
ΔP_1	8.2645×10^{-3}	8.2645×10^{-3}	9.9174×10^{-1}	8.2645×10^{-3}	1.6529×10^{-1}	1.6529×10^{-1}
$\Delta \lambda_1$	4.0000	4.0000	-3.3333×10^{-2}	4.0000	1.8000	1.8000
ΔF_2	3.3058×10^{-2}	-3.3058×10^{-2}	3.3058×10^{-2}	2.9752×10^{-1}	3.3058×10^{-2}	2.9752×10^{-1}
ΔP_2	8.2645×10^{-3}	9.9174×10^{-1}	8.2645×10^{-3}	1.6529×10^{-1}	8.2645×10^{-3}	1.6529×10^{-1}
$\Delta \lambda_2$	4.0000	-3.3333×10^{-2}	4.0000	1.8000	4.0000	1.8000
ΔF_3	0.0000	3.3058×10^{-2}	3.3058×10^{-2}	2.9752×10^{-1}	2.9752×10^{-1}	3.3058×10^{-2}
ΔP_3	1.0000	8.2645×10^{-3}	8.2645×10^{-3}	1.6529×10^{-1}	1.6529×10^{-1}	8.2645×10^{-3}
$\Delta \lambda_3$	0.0000	4.0000	4.0000	1.8000	1.8000	4.0000
$\Delta \lambda$	8.0000	7.9667	7.9667	7.6000	7.6000	7.6000
ΔP	6.8301×10^{-5}	6.7737×10^{-5}	6.7737×10^{-5}	2.2579×10^{-4}	2.2579×10^{-4}	2.2579×10^{-4}
ΔF	5.4641×10^{-4}	5.3964×10^{-4}	5.3964×10^{-4}	1.7160×10^{-3}	1.7160×10^{-3}	1.7160×10^{-3}

3 Demonstration and Validation

The method developed in sections 2.3 and 2.4 will now be demonstrated on the system described in section 2.1 and decomposed in section 2.2.

Table 2 shows the calculations used to find the frequency of crossing the boundary between L -sets and A -sets. This frequency equals the failure frequency of the system. In Table 2, ΔF_j denotes the difference of the cumulative frequencies between the upper and the lower states of arc j . ΔP_j denotes the difference of the cumulative probabilities between the upper and the lower states of arc j . $\Delta \lambda_j$ represents the difference $(\lambda_{sj}^+ - \lambda_{sj}^-)$ as shown in equation (20). ΔP is the probability of the set and ΔF is the difference between the upward and downward frequencies of this set, per equation (23). The sums of the L -set probabilities and frequencies, obtained from the last two rows, gives the system failure probability and frequency as 8.81×10^{-4} and $6.77 \times 10^{-3}/h$ respectively.

To verify the above results, we use an enumeration process, which shows all the points in the state space. In Table 3, “LL” denotes the property of the state: “X” implies that the corresponding state is a boundary point, i.e., an upward transition from this state results in a success state; “XX” implies that this is a failure state but not a boundary point. “PROBABILITY” is the probability of the state, μ is the repair rate at the corresponding point, and “FREQUENCY” is the frequency of crossing the boundary between success and failure states. From the table, first the boundary points are identified; then the boundary points are used to determine the failure frequency of the system. The frequency of encountering success states, as determined by the enumeration approach, is $6.77 \times 10^{-3}/h$. This is also the failure frequency, in the steady state. This result validates the method developed in this paper.

Table 3 also indicates the L -set to which each failure state belongs. For each failure state, the exact state probability is also indicated. The sum of probabilities of all the member states of each L -set provides the set probability, which matches the corresponding entry in Table 2. The total probability of the L -states, 8.81×10^{-4} , also agrees with the result previously obtained.

4 Conclusion

This paper has described a method for calculating the frequency of failure of a discrete capacity system, using the method of state space decomposition. The formulation of the general method is then followed by the description of a special technique for easy implementation. Then an example is provided to illustrate the method and to compare the results with those obtained by state enumeration.

The strength of the method lies in the following features:

- the method of decomposition deals with sets of states rather than individual states, and is therefore more efficient than methods involving enumeration;
- the expressions used for determination of F&D indices are closed-form, unlike previously developed recursive forms;
- the calculation of F&D indices requires only slight additional computational effort beyond the effort required for calculation of probability indices.

While the method of decomposition and calculation of probability indices therefrom is well known, closed form expressions for calculation of F&D indices from decomposition have been presented for the first time in this paper.

Table 3: Determination of frequency of system failure using state enumeration.

	STATE	LL	SET	PROBABILITY	μ	FREQUENCY
1	0,0,0	XX	L_1	5.6447×10^{-7}		
2	0,0,100	XX	L_1	1.1289×10^{-5}		
3	0,0,200	X	L_1	5.6447×10^{-5}	8	4.5158×10^{-4}
4	0,100,0	XX	L_2	1.1289×10^{-5}		
5	0,100,100	X	L_4	2.2579×10^{-4}	8	1.8063×10^{-3}
6	0,100,200					
7	0,200,0	X	L_2	5.6447×10^{-5}	8	4.5158×10^{-4}
8	0,200,100					
9	0,200,200					
10	100,0,0	XX	L_3	1.1289×10^{-5}		
11	100,0,100	X	L_5	2.2579×10^{-4}	8	1.8063×10^{-3}
12	100,0,200					
13	100,100,0	X	L_6	2.2579×10^{-4}	8	1.8063×10^{-3}
14	100,100,100					
15	100,100,200					
16	100,200,0					
17	100,200,100					
18	100,200,200					
19	200,0,0	X	L_3	5.6447×10^{-5}	8	4.5158×10^{-4}
20	200,0,100					
21	200,0,200					
22	200,100,0					
23	200,100,100					
24	200,100,200					
25	200,200,0					
26	200,200,100					
27	200,200,200					
	TOTALS		L	8.8114×10^{-4}		6.7737×10^{-3}

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