Problem 1 (20 points): Using Mason’s rule, determine the transfer function $\frac{Y(s)}{X(s)}$ for the signal flow graph depicted below. Write the final result as a ratio of polynomials such that all powers of $s$ are positive.

\[ \text{# fwd paths: 1} \]
\[ \text{# loops: 4} \]
\[ P_1 = \frac{1}{s^3} \quad \Delta_1 = 1 \]
\[ L_1 = -\frac{1}{s} \]
\[ L_2 = -\frac{2}{s^2} \]
\[ L_3 = \frac{-3}{s} \]
\[ L_4 = -\frac{4}{s^2} \]
\[ \Delta = 1 - (L_1 + L_2 + L_3 + L_4) + L_1 L_3 \]
\[ \Delta = 1 + \frac{\frac{1}{s}}{s} + \frac{\frac{2}{s^2}}{s} + \frac{\frac{3}{s}}{s} + \frac{\frac{4}{s^2}}{s^2} + \frac{\frac{3}{s^2}}{s^2} \]
\[ \Delta = 1 + \frac{4}{s} + \frac{9}{s^2} \]
\[ \frac{Y}{X} = \left( \frac{1}{s^3} \right) \left( 1 + \frac{4}{s} + \frac{9}{s^2} \right) \left( \frac{s^3}{s^3} \right) = \frac{1}{s^3 + 4s^2 + 9s} \]
Problem 2 (20 points): Draw the free body diagram and evaluate the transfer function \( \Theta(s)/T(s) \) for the rotational mechanical system shown below.

\[
T = (Js^2 + 3k + 3bs) \Theta
\]

\[
\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + 3bs + 3k}
\]
Problem 3 (30 points): Draw the free body diagram and compute the transfer function $X_3(s)/F(s)$ for the translational mechanical system shown below.

\[
\begin{align*}
X_3 & : \quad F = s^2X_3 + X_3 - X_1 + X_3 - X_2 \\
X_2 & : \quad 0 = s^2X_2 + X_2 + X_2 - X_3 \\
X_1 & : \quad 0 = s^2X_1 + X_1 + X_1 - X_3
\end{align*}
\]

\[
\begin{bmatrix}
F \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
(s^2+2) & -1 & -1 \\
-1 & (s^2+2) & 0 \\
-1 & 0 & (s^2+2)
\end{bmatrix} \begin{bmatrix}
X_3 \\
X_2 \\
X_1
\end{bmatrix}
\]

\[
X_3 = \frac{\Delta_1}{\Delta} = \frac{1}{\Delta} \begin{vmatrix}
F & -1 & -1 \\
0 & (s^2+2) & 0 \\
0 & 0 & (s^2+2)
\end{vmatrix} = \frac{1}{\Delta} F(s^2+2)^2
\]

\[
\Delta = (s^2+2)^3 - (s^2+2) - (s^2+2) = (s^2+2)^3 - 2(s^2+2)
\]

\[
\frac{X_3}{\varepsilon} = \frac{(s^2+2)^2}{(s^2+2)^3 - 2(s^2+2)} = \frac{s^2+2}{s^4 + 4s^2 + 2}
\]
Problem 4 (15 points): The output of an electrical system provides the input to a mechanical system. The differential equation that describes the behavior of the electrical system is \( x''''(t) + 3x'''(t) + 2x''(t) + 4x'(t) = r'(t) + 2r(t) \), where each ‘prime’ indicates a time derivative. The input of the electrical system is indicated by \( r(t) \), and the output of the electrical system is indicated by \( x(t) \). The differential equation for the mechanical system is \( y''(t) + 4y'(t) + 2y(t) = x'(t) + x(t) \), where the input of the mechanical system is \( x(t) \) and the output is \( y(t) \). The initial conditions are \( r(0) = 1, x''(0) = 1, x'(0) = 2, x(0) = 3, y'(0) = 1 \), and \( y(0) = 2 \). Find the transfer function for the combined system \( Y(s)/R(s) \).

\[
\begin{align*}
\mathcal{L}\{x(t)\} &= \frac{s^3 X(s) + 3s^2 X(s) + 2s X(s) + 4X(s)}{s^3 + 3s^2 + 2s + 4} = sR(s) + 2R(s) \\
\frac{X(s)}{R(s)} &= \frac{s + 2}{s^3 + 3s^2 + 2s + 4} \\

\mathcal{L}\{y(t)\} &= \frac{s^2 Y(s) + 4s Y(s) + 2Y(s)}{s^2 + 4s + 2} = s X(s) + X(s) \\
\frac{Y(s)}{X(s)} &= \frac{s + 1}{s^2 + 4s + 2} \\

\frac{Y(s)}{R(s)} &= \frac{\mathcal{L}\{x(t)\}}{\mathcal{L}\{x(t)\}} \cdot \frac{X(s)}{R(s)} = \frac{s + 1}{s^3 + 3s^2 + 2s + 4} \cdot \frac{s + 2}{s^2 + 3s^2 + 2s + 4} \\

\frac{Y(s)}{R(s)} &= \frac{(s + 1)(s + 2)}{(s^2 + 4s + 2)(s^3 + 3s^2 + 2s + 4)}
\end{align*}
\]
Problem 5 (15 points): Convert the signal flow graph below into a block diagram. Be sure to use the same number of gain blocks in your drawing as branches in the signal flow graph depicted below.