Nonlinear Systems and Control
Lecture # 25

Stabilization

Feedback Lineaization
Consider the nonlinear system

\[ \dot{x} = f(x) + G(x)u \]

\[ f(0) = 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \]

Suppose there is a change of variables \( z = T(x) \), defined for all \( x \in D \subset \mathbb{R}^n \), that transforms the system into the controller form

\[ \dot{z} = Az + B\gamma(x)[u - \alpha(x)] \]

where \((A, B)\) is controllable and \( \gamma(x) \) is nonsingular for all \( x \in D \)

\[ u = \alpha(x) + \gamma^{-1}(x)v \quad \Rightarrow \quad \dot{z} = Az + Bv \]
\[ v = -Kz \]

Design \( K \) such that \((A - BK)\) is Hurwitz

The origin \( z = 0 \) of the closed-loop system

\[
\dot{z} = (A - BK)z
\]

is globally exponentially stable

\[
 u = \alpha(x) - \gamma^{-1}(x)KT(x)
\]

Closed-loop system in the \( x \)-coordinates:

\[
\dot{x} = f(x) + G(x) \left[ \alpha(x) - \gamma^{-1}(x)KT(x) \right]
\]
What can we say about the stability of $x = 0$ as an equilibrium point of

$$\dot{x} = f(x) + G(x) \left[ \alpha(x) - \gamma^{-1}(x)KT(x) \right]$$

$x = 0$ is asymptotically stable because $T(x)$ is a diffeomorphism. Show it!

Is $x = 0$ globally asymptotically stable? In general No

It is globally asymptotically stable if $T(x)$ is a global diffeomorphism (See page 508)
What information do we need to implement the control

\[ u = \alpha(x) - \gamma^{-1}(x)KT(x) \]

What is the effect of uncertainty in \( \alpha, \gamma, \) and \( T? \)

Let \( \hat{\alpha}(x), \hat{\gamma}(x), \) and \( \hat{T}(x) \) be nominal models of \( \alpha(x), \gamma(x), \) and \( T(x) \)

\[ u = \hat{\alpha}(x) - \hat{\gamma}^{-1}(x)K\hat{T}(x) \]

Closed-loop system:

\[ \dot{z} = (A - BK)z + B\delta(z) \]

\[ \delta = \gamma[\hat{\alpha} - \alpha + \gamma^{-1}KT - \hat{\gamma}^{-1}K\hat{T}] \]
\[ \dot{z} = (A - BK)z + B\delta(z) \quad (\*) \]

\[ V(z) = z^T P z, \quad P(A - BK) + (A - BK)^T P = -I \]

Lemma 13.3

- If \[ \|\delta(z)\| \leq k\|z\| \] for all \( z \), where

\[ 0 \leq k < \frac{1}{2\|PB\|} \]

then the origin of (\*) is globally exponentially stable.

- If \[ \|\delta(z)\| \leq k\|z\| + \varepsilon \] for all \( z \), then the state \( z \) is globally ultimately bounded by \( \varepsilon c \) for some \( c > 0 \).
Example (Pendulum Equation):

\[
\ddot{\theta} = -a \sin \theta - b \dot{\theta} + cT
\]

\[
x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{ss} = T - \frac{a}{c} \sin \delta
\]

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin \delta] - bx_2 + cu
\end{align*}
\]

\[
u = \frac{1}{c} \left\{ a[\sin(x_1 + \delta) - \sin \delta] - k_1 x_1 - k_2 x_2 \right\}
\]

\[
A - BK = \begin{bmatrix}
0 & 1 \\
-k_1 & -(k_2 + b)
\end{bmatrix} \text{ is Hurwitz}
\]
\[ T = u + \frac{a}{c} \sin \delta = \frac{1}{c} [a \sin(x_1 + \delta) - k_1 x_1 - k_2 x_2] \]

Let \( \hat{a} \) and \( \hat{c} \) be nominal models of \( a \) and \( c \)

\[ T = \frac{1}{\hat{c}} [\hat{a} \sin(x_1 + \delta) - k_1 x_1 - k_2 x_2] \]

\[ \dot{x} = (A - BK)x + B\delta(x) \]

\[ \delta(x) = \left( \frac{\hat{a}c - a\hat{c}}{\hat{c}} \right) \sin(x_1 + \delta_1) - \left( \frac{c - \hat{c}}{\hat{c}} \right) (k_1 x_1 + k_2 x_2) \]
\[
\delta(x) = \left( \frac{\hat{a}c - a\hat{c}}{\hat{c}} \right) \sin(x_1 + \delta_1) - \left( \frac{c - \hat{c}}{\hat{c}} \right) (k_1x_1 + k_2x_2)
\]

\[
|\delta(x)| \leq k\|x\| + \varepsilon
\]

\[
k = \left| \frac{\hat{a}c - a\hat{c}}{\hat{c}} \right| + \left| \frac{c - \hat{c}}{\hat{c}} \right| \sqrt{k_1^2 + k_2^2}, \quad \varepsilon = \left| \frac{\hat{a}c - a\hat{c}}{\hat{c}} \right| |\sin \delta_1|
\]

\[
P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}
\]

\[
k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}
\]

\[
\sin \delta_1 = 0 \Rightarrow \varepsilon = 0
\]
Is feedback linearization a good idea?

Example

\[ \dot{x} = ax - bx^3 + u, \quad a, b > 0 \]

\[ u = -(k + a)x + bx^3, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx \]

\(-bx^3\) is a damping term. Why cancel it?

\[ u = -(k + a)x, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx - bx^3 \]

Which design is better?
Example

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -h(x_1) + u \]

\[ h(0) = 0 \text{ and } x_1 h(x_1) > 0, \quad \forall \ x_1 \neq 0 \]

Feedback Linearization:

\[ u = h(x_1) - (k_1 x_1 + k_2 x_2) \]

With \( y = x_2 \), the system is passive with

\[ V = \int_0^{x_1} h(z) \, dz + \frac{1}{2} x_2^2 \]

\[ \dot{V} = h(x_1) \dot{x}_1 + x_2 \dot{x}_2 = y u \]
The control

\[ u = -\sigma(x_2), \quad \sigma(0) = 0, \quad x_2 \sigma(x_2) > 0 \quad \forall \ x_2 \neq 0 \]

creates a feedback connection of two passive systems with storage function \( V \)

\[ \dot{V} = -x_2 \sigma(x_2) \]

\[ x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0 \]

Asymptotic stability of the origin follows from the invariance principle

Which design is better? (Read Example 13.20)