Nonlinear Systems and Control
Lecture # 15
Positive Real Transfer Functions
&
Connection with Lyapunov Stability
Definition: A $p \times p$ proper rational transfer function matrix $G(s)$ is positive real if

- poles of all elements of $G(s)$ are in $Re[s] \leq 0$
- for all real $\omega$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega} (s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is called strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$
Scalar Case ($p = 1$):

$$G(j\omega) + G^T(-j\omega) = 2 \text{Re}[G(j\omega)]$$

$\text{Re}[G(j\omega)]$ is an even function of $\omega$. The second condition of the definition reduces to

$$\text{Re}[G(j\omega)] \geq 0, \ \forall \ \omega \in [0, \infty)$$

which holds when the Nyquist plot of $G(j\omega)$ lies in the closed right-half complex plane

This is true only if the relative degree of the transfer function is zero or one
Lemma: A $p \times p$ proper rational transfer function matrix $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega) + G^T(-j\omega) > 0$, $\forall \omega \in \mathbb{R}$
- $G(\infty) + G^T(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^{2(p-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$$

where $q = \text{rank}[G(\infty) + G^T(\infty)]$
Scalar Case ($p = 1$): $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $\text{Re}[G(j\omega)] > 0$, $\forall \omega \in [0, \infty)$
- $G(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^2 \text{Re}[G(j\omega)] > 0$$
Example:

\[ G(s) = \frac{1}{s} \]

has a simple pole at \( s = 0 \) whose residue is 1

\[ Re[G(j\omega)] = Re \left[ \frac{1}{j\omega} \right] = 0, \quad \forall \omega \neq 0 \]

Hence, \( G \) is positive real. It is not strictly positive real since

\[ \frac{1}{(s - \varepsilon)} \]

has a pole in \( Re[s] > 0 \) for any \( \varepsilon > 0 \)
Example:

\[ G(s) = \frac{1}{s + a}, \quad a > 0, \quad \text{is Hurwitz} \]

\[ \text{Re}[G(j\omega)] = \frac{a}{\omega^2 + a^2} > 0, \quad \forall \ \omega \in [0, \infty) \]

\[ \lim_{\omega \to \infty} \omega^2 \text{Re}[G(j\omega)] = \lim_{\omega \to \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0 \ \Rightarrow \ G \text{ is SPR} \]

Example:

\[ G(s) = \frac{1}{s^2 + s + 1}, \quad \text{Re}[G(j\omega)] = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2} \]

G is not PR
Example:

\[ G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix} \text{ is Hurwitz} \]

\[ G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & -\frac{2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} \geq 0, \quad \forall \omega \in \mathbb{R} \]

\[ G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = 1 \]

\[ \lim_{\omega \to \infty} \omega^2 \det[G(j\omega) + G^T(-j\omega)] = 4 \quad \Rightarrow \quad G \text{ is SPR} \]
Positive Real Lemma: Let

\[ G(s) = C(sI - A)^{-1}B + D \]

where \((A, B)\) is controllable and \((A, C)\) is observable. \(G(s)\) is positive real if and only if there exist matrices \(P = P^T > 0, L,\) and \(W\) such that

\[
\begin{align*}
PA + A^TP &= -L^TL \\
PB &= C^T - L^TW \\
W^TW &= D + D^T
\end{align*}
\]
Kalman–Yakubovich–Popov Lemma: Let

\[ G(s) = C(sI - A)^{-1}B + D \]

where \((A, B)\) is controllable and \((A, C)\) is observable. \(G(s)\) is strictly positive real if and only if there exist matrices \(P = P^T > 0\), \(L\), and \(W\), and a positive constant \(\varepsilon\) such that

\[
\begin{align*}
PA + A^TP &= -L^TL - \varepsilon P \\
PB &= C^T - L^TW \\
W^TW &= D + D^T
\end{align*}
\]
Lemma: The linear time-invariant minimal realization

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

with

\[ G(s) = C(sI - A)^{-1}B + D \]

is

- passive if \( G(s) \) is positive real
- strictly passive if \( G(s) \) is strictly positive real

Proof: Apply the PR and KYP Lemmas, respectively, and use \( V(x) = \frac{1}{2}x^TPx \) as the storage function
\[
\begin{aligned}
  u^T y - \frac{\partial V}{\partial x}(Ax + Bu) \\
  = u^T (Cx + Du) - x^T P(Ax + Bu) \\
  = u^T Cx + \frac{1}{2} u^T (D + D^T)u \\
  - \frac{1}{2} x^T (PA + A^T P)x - x^T PBu \\
  = u^T (B^T P + W^T L)x + \frac{1}{2} u^T W^T W u \\
  + \frac{1}{2} x^T L^T L x + \frac{1}{2} \varepsilon x^T P x - x^T PBu \\
  = \frac{1}{2} (Lx + Wu)^T (Lx + Wu) + \frac{1}{2} \varepsilon x^T P x \
\end{aligned}
\]

In the case of the PR Lemma, \( \varepsilon = 0 \), and we conclude that the system is passive; in the case of the KYP Lemma, \( \varepsilon > 0 \), and we conclude that the system is strictly passive.
Connection with Lyapunov Stability

**Lemma:** If the system

\[ \dot{x} = f(x, u), \quad y = h(x, u) \]

is passive with a positive definite storage function \( V(x) \), then the origin of \( \dot{x} = f(x, 0) \) is stable

**Proof:**

\[ u^T y \geq \frac{\partial V}{\partial x} f(x, u) \Rightarrow \frac{\partial V}{\partial x} f(x, 0) \leq 0 \]
Lemma: If the system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is strictly passive, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Proof: The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \psi(x) \Rightarrow \frac{\partial V}{\partial x} f(x, 0) \leq -\psi(x)$$

Why is $V(x)$ positive definite? Let $\phi(t; x)$ be the solution of $\dot{z} = f(z, 0), \ z(0) = x$
\[ \dot{V} \leq -\psi(x) \]

\[ V(\phi(\tau, x)) - V(x) \leq -\int_0^\tau \psi(\phi(t; x)) \, dt, \quad \forall \tau \in [0, \delta] \]

\[ V(\phi(\tau, x)) \geq 0 \implies V(x) \geq \int_0^\tau \psi(\phi(t; x)) \, dt \]

\[ V(\bar{x}) = 0 \implies \int_0^\tau \psi(\phi(t; \bar{x})) \, dt = 0, \quad \forall \tau \in [0, \delta] \]

\[ \implies \psi(\phi(t; \bar{x})) \equiv 0 \implies \phi(t; \bar{x}) \equiv 0 \implies \bar{x} = 0 \]
Definition: The system
\[ \dot{x} = f(x, u), \quad y = h(x, u) \]
is zero-state observable if no solution of \( \dot{x} = f(x, 0) \) can stay identically in \( S = \{ h(x, 0) = 0 \} \), other than the zero solution \( x(t) \equiv 0 \)

Linear Systems
\[ \dot{x} = Ax, \quad y = Cx \]
Observability of \((A, C)\) is equivalent to
\[ y(t) = Ce^{At}x(0) \equiv 0 \iff x(0) = 0 \iff x(t) \equiv 0 \]
Lemma: If the system

\[ \dot{x} = f(x, u), \quad y = h(x, u) \]

is output strictly passive and zero-state observable, then the origin of \( \dot{x} = f(x, 0) \) is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Proof: The storage function \( V(x) \) is positive definite

\[ u^T y \geq \frac{\partial V}{\partial x} f(x, u) + y^T \rho(y) \Rightarrow \frac{\partial V}{\partial x} f(x, 0) \leq -y^T \rho(y) \]

\[ \dot{V}(x(t)) \equiv 0 \Rightarrow y(t) \equiv 0 \Rightarrow x(t) \equiv 0 \]

Apply the invariance principle
Example

\[\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -ax_1^3 - kx_2 + u, \\
y &= x_2, \\
a, k &> 0
\end{align*}\]

\[V(x) = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2\]

\[\dot{V} = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu\]

The system is output strictly passive

\[y(t) \equiv 0 \iff x_2(t) \equiv 0 \implies ax_1^3(t) \equiv 0 \implies x_1(t) \equiv 0\]

The system is zero-state observable. \(V\) is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable.