• Claim (Proof of Theorem 4.1, page 116): By repetition of of previous arguments, we know that for every $a > 0$, we can choose $b > 0$ such that $\Omega_b \subset B_a$.

Proof: It is enough to consider $a < r$. Let $\gamma = \min_{a \leq \|x\| \leq r} V(x)$ and take $b < \gamma$. Then, $\Omega_b = \{x \in B_r \mid V(x) \leq b\}$ is in the interior of $B_a$. If this was not the case, there would be a point $p \in \Omega_b$ that lies in the region $a \leq \|x\| \leq r$. At this point $V(p) \geq \gamma > b$, which contradicts that fact that for $x \in \Omega_a$, $V(x) \leq a < \gamma$.

• Claim (page 662): $\psi(s)$ is continuous, positive definite, and increasing. There is a class $\mathcal{K}$ function $\alpha_1(s)$ such that $\alpha_1(s) \leq k\psi(s)$ with $0 < k < 1$.

Proof: Take
\[
\alpha_1(s) = \frac{ks}{s+1} \psi(s), \quad \text{for } s \geq 0
\]
$\alpha_1$ is strictly increasing because $s/(s+1)$ is strictly increasing and $\psi$ is increasing and positive.
\[
\frac{s}{s+1} \leq 1 \implies \alpha_1(s) \leq k\psi(s)
\]

• Claim (page 664): $\delta(\varepsilon)$ is positive definite, nondecreasing, but not necessarily continuous. There is a class $\mathcal{K}$ function $\zeta(r)$ such that $\zeta(r) \leq k\delta(r)$ with $0 < k < 1$.

Proof: Note from the last line of page 663 that $\delta(\varepsilon) \leq \varepsilon$. Let
\[
\zeta(s) = k \int_0^s e^{-\sigma} \tilde{\delta}(\sigma) \ d\sigma
\]
Because $\tilde{\delta}$ is monotone, it is Riemann integrable. Because the product of two Riemann integrable functions is Riemann integrable, $e^{-\sigma} \tilde{\delta}(\sigma)$ is Riemann integrable.
\[
\zeta(s) - \zeta(r) = k \int_r^s e^{-\sigma} \tilde{\delta}(\sigma) \ d\sigma
\]
$\zeta$ is continuous because
\[
|\zeta(s) - \zeta(r)| \leq k\varepsilon|s-r|
\]
$\zeta$ is strictly increasing because $e^{-\sigma} \tilde{\delta}(\sigma) > 0$ for all $\sigma > 0$. Finally, because $\tilde{\delta}$ is nondecreasing
\[
\tilde{\delta}(\sigma) \leq \tilde{\delta}(s^-) \leq \tilde{\delta}(s) \leq \tilde{\delta}(s^+), \quad \text{for } 0 < \sigma < s
\]
where $\tilde{\delta}(s^-)$ and $\tilde{\delta}(s^+)$ are the left and right limits of $\tilde{\delta}$ at $s$.
\[
\zeta(s) \leq k \int_0^s e^{-\sigma} \tilde{\delta}(s) \ d\sigma = k(1 - e^{-s})\tilde{\delta}(s) \leq k\tilde{\delta}(s)
\]