Chapter 3: Transformers (Textbook Chapter 2)

Chapter Objectives
In this chapter you will be able to:
• Choose the correct rating and characteristics of a transformer for a specific application
• Calculate the losses, efficiency, and voltage regulation of a transformer under specific operating conditions.
• Experimentally determine the transformer parameters given its ratings.
• Utilize the per unit system.

3.1 Introduction
Transformers do not have moving parts, nor are they energy conversion devices, however their ability to modify the current-voltage characteristics of a given load or source, make them invaluable components in energy conversion systems. They are utilized for power applications and in low power signal processing systems. One application in power transmission is the use of transformers on a transmission line utility pole commonly seen as a cylinder with a few wires sticking out. These wires enter the transformer through bushings that provide isolation between the wires and the tank. Inside the tank there is an iron core commonly made of silicon-steel laminations that are 14 mils (0.014”) thick. The insulation often used is paper with the whole coil system immersed in insulating oil. The oil increases the dielectric strength of the paper and helps to transfer heat from the core/coil assembly. An drawing of one such distribution transformer is shown in Figure 2.2 in your textbook. Connection of the transformer to the transmission lines can take several electrical configurations. A relatively simple connection to a 2.4 kV three phase transmission line is shown in Figure 1.

Figure 1. Example configuration of a distribution pole transformer connection to three phase power lines to provide 120 (V) service to your home.
Figure 2. Example configuration of a distribution pole transformer connection to three phase power lines and provides 120 (V) and 240 (V) service.

If a neutral line is also part of the three phase transmission line (perhaps between the substation and your home), then the connection could be made as shown in Figure 3.

Figure 3. Distribution transformer connection to provide 120 (V) and 240 (V) service from a 4160Y/2400 (V) four-wire transmission line.
For a three phase line at the service end, a system could be connected to a four wire three phase transmission line source as shown in Figure 4 below.

Figure 4. Three phase to three phase distribution transformer connection providing a four-wire three phase distribution of 120 (V) and 208 (V) service from a 4160Y/2400 (V) four-wire transmission line.

Transformers are very common place in society, and have had significant impact at many power levels. They also continue to be improved on today, with such studies as amorphous metals to further decrease core losses. For more understanding, we begin with the ideal transformer.

3.2 Ideal Transformer

Under ideal conditions, the iron core has infinite permeability and the coils have zero electrical resistance. Coil 1 has $N_1$ turns, and coil 2 has $N_2$ turns, and all of the magnetic flux is maintained in the iron (no flux leakage).

Figure 5. Transformer and equivalent magnetic circuit.
The electromotive force, or emf, is represented with the symbol \( e \). For an ideal transformer, this is the voltage at the terminals of a given coil. The flux linkage in each coil is \( \lambda_1 = N_1 \phi \) and \( \lambda_2 = N_2 \phi \). The electromotive force induced in each coil is then the time derivative of the flux linkage or

\[
e_1 = \frac{d\lambda_1}{dt} = N_1 \frac{d\phi}{dt} \text{ (V)} \tag{3.1}
\]

\[
e_2 = \frac{d\lambda_2}{dt} = N_2 \frac{d\phi}{dt} \text{ (V)} \tag{3.2}
\]

The ratio of the voltage at the terminals of coil 1 to the voltage at coil 2 is then

\[
\frac{e_1}{e_2} = \frac{N_1}{N_2} \tag{3.3}
\]

Using an equivalent circuit shown in Figure 5 with the magnetomotive force \( F_1 \) for coil 1 and \( F_2 \) for coil 2 we would write:

\[
F_1 - F_2 = N_1 i_1 - N_2 i_2 = 0 \tag{3.4}
\]

or

\[
\frac{i_1}{i_2} = \frac{N_2}{N_1} \tag{3.5}
\]

Transformers are often used with a voltage source connected to one coil and a load connected to the other coil. The dots above the coils help to indicate the voltage reference marks for the coils such that for an ideal transformer a positive voltage, \( v_1 = e_1 = N_1 \frac{d\phi}{dt} \) in coil 1 (primary coil) results in a positive voltage, \( v_2 = e_2 = N_2 \frac{d\phi}{dt} \) in the coil 2 (secondary coil) as shown in Figure 6.

![Figure 6. Circuit utilizing a transformer.](image-url)
The circuit symbol for the transformer is shown in Figure 7. Since the voltages across the coils transform in the direct ratio of the turns and the current transforms in the inverse ratio of the turns, then the impedance can also be transformed through the transformer.

\[ N_1 \quad \text{i}_1 \quad \text{i}_2 \quad N_2 \]

Figure 7. Transformer circuit symbol.

For a voltage applied to the primary coil, and a load impedance, \( Z_L \), connected to the secondary as shown in Figure 8.

\[ \hat{V}_1 = \frac{N_1}{N_2} \hat{V}_2 \quad (3.6) \]

\[ \hat{I}_1 = \frac{N_2}{N_1} \hat{I}_2 \quad (3.7) \]

Then the impedance measured at the terminals of the primary is

\[ \frac{\hat{V}_1}{\hat{I}_1} = Z_1 = \left( \frac{N_2}{N_1} \right)^2 \frac{\hat{V}_2}{\hat{I}_2} \quad (3.8) \]

Thus the following circuits are equivalent.
3.3 Non-Ideal Transformer

There are several non-ideal properties that we can take into account by modifying the equivalent circuit shown in Figure 7. These include a finite core permeability ($\mu_c < \infty$), leakage flux, core losses, and coil resistance. The contribution from each of these is discussed below.

**Finite Core Permeability**

For the core of the transformer to have a finite permeability, then the circuit in Figure 5 is modified to include the reluctance of the core.

Then we can write

$$F_1 - F_2 = N_1i_1 - N_2i_2 = R\phi$$  \hspace{1cm} (3.9)

Defining a magnetomotive force that is equal to the drop across the reluctance of the core as:

$$F_c = N_1i_{1,ex} = R\phi$$  \hspace{1cm} (3.10)

Solving for the flux gives

$$\phi = \frac{N_1i_{1,ex}}{R}$$  \hspace{1cm} (3.11)

The rate of change in the flux is proportional to the induced voltage as
\[
e_1 = N_1 \frac{d\phi}{dt} = N_1 \frac{d}{dt} \left( \frac{N_1 i_{1,ex}}{R} \right) = \left( \frac{N_1^2}{R} \right) \frac{di_{1,ex}}{dt}
\]

(3.12)

This induced voltage is proportional to the time derivative of the current \(i_{1,ex}\) which can be represented by an inductor in our equivalent circuit with a value of \(L_m = \frac{N_1^2}{R}\). The equivalent circuit for the ideal transformer is then modified to account for the finite permeability of the core by placing an additional inductor across the primary coil as shown in Figure 11.

![Equivalent circuit for a transformer with finite core permeability.](image)

Figure 11. Equivalent circuit for a transformer with finite core permeability.

We can also see from Figure 11 that \(i_1 - i_{1,ex} = i'_1\) which is in agreement with equations (3.9) and (3.10).

**Leakage Flux**

With finite core permeability, not all of the flux will be confined to the metal core, but some will “leak” outside the core in the surrounding air. The influence of this leakage flux can also be included in the equivalent circuit, by considering an additional reluctance associated with the leakage flux, \(\phi_l\), for coil 1, and \(\phi_l\) for coil 2 such that

\[
\phi_{l1} = \frac{N_1 i_1}{R_l} \quad \phi_{l2} = \frac{N_2 i_2}{R_l}
\]

(3.13)

These reduce the magnetizing flux, \(\phi_m\), of the core, and modify the induced voltages for the primary (coil 1) and secondary (coil 2) to be

\[
v_1 = \frac{d\phi_{l1}}{dt} = N_1 \left( \frac{d\phi_m}{dt} \right) + N_1 \left( \frac{d\phi_{l1}}{dt} \right) = e_1 + \left( \frac{N_1^2}{R_l} \right) \frac{di_1}{dt}
\]

(3.14)

\[
v_2 = \frac{d\phi_{l2}}{dt} = N_2 \left( \frac{d\phi_m}{dt} \right) + N_2 \left( \frac{d\phi_{l2}}{dt} \right) = e_2 + \left( \frac{N_2^2}{R_l} \right) \frac{di_2}{dt}
\]
The last term for $v_1$ can be represented by an inductor, $L_{l1} = \left( \frac{N_1^2}{R_{l1}} \right)$ and the last term for $v_2$ can be represented by $L_{l2} = \left( \frac{N_2^2}{R_{l2}} \right)$. These can be incorporated into the equivalent circuit as shown in Figure 12.

![Figure 12](image)

*Figure 12. Equivalent circuit for a transformer with finite core permeability and leakage flux.*

**Core Losses**

The dissipation of heat in the core due to the flux through it can be represented by a resistor in the equivalent circuit. It is placed in parallel to $L_m$ since the power loss in the core is proportional to the flux through the core squared, and thus is proportional to $\mathcal{E}_1^2$.

**Resistance of the Coils**

The resistance of copper is approximately $16.8 \times 10^{-9} \, (\Omega \cdot m)$, however with hundreds to thousands of turns for the primary and secondary coils it can lead to an appreciable resistance. Losses to these resistances are proportional to the current through the coils squared.

Adding the core losses and resistance of the coils to the equivalent circuit we obtain our completed model of the transformer as shown in Figure 13.

![Figure 13](image)

*Figure 13. Equivalent circuit for a non-ideal transformer.*
Example

Consider a transformer with a turns ratio of $4000/120$, with a primary coil resistance $R_1 = 1.6$ (Ω), a secondary coil resistance of $R_2 = 1.44$ (mΩ), leakage flux that corresponds to $L_{l1} = 21$ (mH), and $L_{l2} = 19$ (µH), and a realistic core characterized by $R_c = 160$ (kΩ) and $L_m = 450$ (H). The low voltage side of the transformer is at $60$ (Hz), and $V_2 = 120$ (V), and the power there is $P_2 = 20$ (kW) at $pf = 0.85$ lagging. Calculate the voltage at the high voltage side and the efficiency of the transformer.

$$X_m = \omega L_m = 2\pi 60 \cdot 450 = 169.7 \text{ (kΩ)}$$
$$X_1 = \omega L_{l1} = 2\pi 60 \cdot 0.021 = 7.92 \text{ (Ω)}$$
$$X_2 = \omega L_{l2} = 2\pi 60 \cdot 19 \times 10^{-6} = 7.16 \text{ (mΩ)}$$
$$\hat{I}_2 = \frac{P_2}{(V_2 \cdot pf)\angle -31.8^\circ} = 196.08\angle -31.8^\circ \text{ (A)}$$
$$\hat{E}_2 = \hat{V}_2 + \hat{I}_2 (R_2 + jX_2) = 120.98 + j1.045 \text{ (V)}$$
$$\hat{E}_1 = \left(\frac{N_1}{N_2}\right)\hat{E}_2 = 4032.7 + j34.83 = 4032.8/0.49^\circ \text{ (V)}$$
$$\hat{I}_1 = \left(\frac{N_2}{N_1}\right)\hat{I}_2 = 5.001 - j3.1017 \text{ (A)}$$
$$\hat{I}_{1,ex} = \frac{\hat{E}_1}{R_c} + \frac{1}{jX_m} = 0.0254 - j0.0236 \text{ (A)}$$
$$\hat{I}_1 = \hat{I}_{1,ex} + \hat{I}_1 = 5.0255 - j3.125 = 5.918/31.87^\circ \text{ (A)}$$
$$\hat{V}_1 = \hat{E}_1 + \hat{I}_1 (R_1 + jX_1) = 4065.5 + j69.2 \text{ (V)} = 4066/0.9^\circ \text{ (V)}$$

The power losses in the windings and core are:

$$P_{R2} = I_2^2 R_2 = 196.08^2 \cdot 0.0144 = 55.39 \text{ (W)}$$
$$P_{R1} = I_1^2 R_1 = 5.918^2 \cdot 1.6 = 56.04 \text{ (W)}$$
$$P_c = \frac{E_1^2}{R_c} = \frac{4032.8^2}{160 \times 10^3} = 101.64 \text{ (W)}$$
$$P_{loss} = P_{R1} + P_{R2} + P_c = 213.08 \text{ (W)}$$
$$\eta = \frac{P_{out}}{P_{in}} = \frac{P_2}{(P_2 + P_{loss})} = \frac{20 \times 10^3}{20 \times 10^3 + 213.08} = 0.9895$$

3.4 Losses and Ratings

The impedances shown in Figure 13 can be reflected into either the primary side, or the secondary side of the transformer as shown in Figure 14.
Figure 14. Reflection of the impedances to the primary or secondary side of the transformer.

For a given frequency, the power losses in the core (iron losses) increase with the voltage $e_1$ (or $e_2$). If these losses exceed a particular limit, a hot spot in the transformer will reach a temperature that dramatically reduces the life of the insulation. Limits are therefore put on $E_1$ and $E_2$ which are the voltage limits for the transformer. The current limits for the transformer limit $I_1$ and $I_2$ to avoid excessive Joule heating due to winding resistances.

Typically a transformer is described by its rated voltages, $E_{1N}$ and $E_{2N}$, that give both the limits and turns ratio. The ratio of the rated currents $\frac{I_{1N}}{I_{2N}}$, is the inverse of the ratio of the voltages when the magnetizing current is neglected. Instead of listing these rated current limits, a transformer is described by its rated apparent power as:

$$S_N = E_{1N}I_{1N} = E_{2N}I_{2N} \quad (3.15)$$

When a transformer is operated at its rated conditions, that is at its maximum current and maximum voltage, the magnetizing current, $I_{1m}$, typically does not exceed 1% of the current in the transformer. Its effect on the voltage drop on the leakage inductance and on the winding resistance is therefore negligible.

Under maximum (rated) current, the total voltage drops on the winding resistances, and leakage inductances typically do not exceed 6% of the rated voltage. Their effect on $E_1$ and $E_2$ is small and their effect on the magnetizing current can be neglected.

Because these effects are small, we can modify the equivalent circuits shown in Figure 14 to a slightly inaccurate, but much more useful on shown in Figure 15.
Figure 15. Slightly inaccurate, but highly simplified equivalent circuits for the transformer.

When we utilize this simplification and work with the reflected voltages, the transformer equivalent circuits can be shown as:

Figure 16. Working with $V'_2$ or $V'_1$ the above approximate circuits for the transformers can simplify the analysis.
Example (repeat with simplified equivalent circuits)

Consider a transformer with a turns ratio of 4000/120, with a primary coil resistance \( R_1 = 1.6 \) (Ω), a secondary coil resistance of \( R_2 = 1.44 \) (mΩ), leakage flux that corresponds to \( L_{l1} = 21 \) (mH), and \( L_{l2} = 19 \) (µH), and a realistic core characterized by \( R_c = 160 \) (kΩ) and \( L_m = 450 \) (H). The low voltage side of the transformer is at 60 (Hz), and \( V_2 = 120 \) (V), and the power there is \( P_2 = 20 \) (kW) at \( pf = 0.85 \) lagging. Calculate the voltage at the high voltage side and the efficiency of the transformer.

Using the simplified equivalent circuits of Figure 16, we can first find \( R_{eq} \) and \( X_{eq} \)

\[
R_{eq} = R_1 + \left( \frac{N_1}{N_2} \right)^2 R_2 = 3.2 \text{ (Ω)}
\]

\[
X_{eq} = 2\pi \cdot 60 \left( L_{l1} + \left( \frac{N_1}{N_2} \right)^2 L_{l2} \right) = 15.876 \text{ (Ω)}
\]

then

\[
\hat{I}_2 = \frac{P_2}{(V_2 \cdot pf)} = 196.08/-31.8^\circ \text{ (A)}
\]

\[
\hat{E}_2 = \hat{V}_2
\]

\[
\hat{I}'_2 = \hat{I}_2 \left( \frac{N_2}{N_1} \right) = 5 - j3.102 = 5.884/-31.8^\circ \text{ (A)}
\]

\[
\hat{E}_1 = \frac{\hat{E}_2}{\left( \frac{N_1}{N_2} \right)} = 4000 \text{ (V)}
\]

\[
\hat{I}_{1,ex} = \hat{E}_1 \left( \frac{1}{R_c} + \frac{1}{jX_m} \right) = 0.0258 - j0.00235 \text{ (A)}
\]

\[
\hat{I}_1 = \hat{I}_{1,ex} + \hat{I}'_2 = 5.0259 - j3.125 = 5.918/-31.87^\circ \text{ (A)}
\]

\[
\hat{V}_1 = \hat{E}_1 + \hat{I}_1 (R_{eq} + jX_{eq}) = 4065 + j69.79 \text{ (V)} = 4066/0.98^\circ \text{ (V)}
\]

The power losses in the windings and core are:

\[
P_{Req} = I_1^2 R_{eq} = 5.884^2 \cdot 3.2 = 110.79 \text{ (W)}
\]

\[
P_c = \frac{V_1^2}{R_c} = \frac{4065^2}{160 \times 10^3} = 103.28 \text{ (W)}
\]

\[
P_{loss} = P_{Req} + P_c = 214.07 \text{ (W)}
\]

\[
\eta = \frac{P_{out}}{P_{in}} = \frac{P_2}{(P_2 + P_{loss})} = \frac{20 \times 10^3}{20 \times 10^3 + 214.07} = 0.9894
\]

These values agree well with the previous analysis using the more accurate model.
3.4 Per Unit System

A simplification in the analysis can come from expressing each of the values as a fraction of a defined base system of quantities. When this is done, simple problems can be made more complex, however more complex problems can be made easier to solve. As an example consider a simple problem of a load impedance of $10 + j5 \, (\Omega)$ that has a voltage of $100 \, (V)$ connected to it. Calculate the power at the load.

The traditional solution is found as:

$$\hat{I}_L = \frac{\hat{V}_L}{Z_L} = \frac{100}{10 + j5} = 8.94 \angle -26.57^\circ \, (A)$$

and the power is

$$P_L = V_L \cdot I_L \cdot pf = 100 \cdot 8.94 \cdot \cos(26.57^\circ) = 800 \, (W)$$

Using the per unit system to find the solution:

1. First define a consistent system of values for the base. If we choose $V_b = 50 \, (V)$, $I_b = 10 \, (A)$, then $Z_b = V_b/I_b = 5 \, (\Omega)$, and $P_b = V_b^2/I_b = 500 \, (W)$, $Q_b = 500 \, (VAr)$, and $S_b = 500 \, (VA)$.
2. Convert all units to pu: $V_{L,pu} = V_L/V_b = 2pu$, $Z_{L,pu} = (10 + j5)/5 = 2 + j1 \, (pu)$
3. Solve the problem using the pu system:

$$\hat{I}_{L,pu} = \frac{\hat{V}_{L,pu}}{Z_{L,pu}} = \frac{2}{2 + j1} = 0.894 \angle -26.57^\circ \, (pu)$$

$$P_{L,pu} = V_{L,pu} \cdot I_{L,pu} \cdot pf = 2 \cdot 0.894 \cdot \cos(26.57^\circ) = 1.6 \, (pu)$$

4. Convert back to the SI system

$$I_L = I_{L,pu} \cdot I_b = 0.894 \cdot 10 = 8.94 \, (A)$$

$$P_L = P_{L,pu} \cdot P_b = 1.6 \cdot 500 = 800 \, (W)$$

For the more complicated example of a transformer we choose the bases for each side of the transformer such that

$$\begin{align*}
\frac{V_{1b}}{V_{2b}} &= \frac{N_1}{N_2} \\
\frac{I_{1b}}{I_{2b}} &= \frac{N_2}{N_1}
\end{align*}$$

This leads to the two base apparent powers being equal

$$S_{1b} = V_{1b}I_{1b} = V_{2b}I_{2b} = S_{2b}$$

The bases are often chosen to be the rated quantities of the transformer on each side. This is convenient since most of the time transformers operate at rated voltage (making the pu voltage unity), and the currents and power are seldom above rated (seldom above 1 pu).

The base impedances are related by:
To move impedances from one side of the transformer to the other, they get multiplied or divided by the square of the turns ratio, \( \left( \frac{N_2}{N_1} \right)^2 \), but so does the base impedance, hence the pu value of an impedance stays the same regardless of which side of the transformer it is on.

Through our choice of the bases in (3.16) and (3.17), we also see that for an ideal transformer

\[
\begin{align*}
E_{1,pu} &= E_{2,pu} \\
I_{1,pu} &= I_{2,pu}
\end{align*}
\]

Thus when using the per unit system, an ideal transformer has voltages and currents on one side that are identical to the voltages and currents on the other side, and the ideal transformer can be eliminated.

**Example (repeat and solved using pu system)**

Consider a 30 (kVA) rated transformer with a turns ratio of 4000/120, with a primary coil resistance \( R_1 = 1.6 \) (\( \Omega \)), a secondary coil resistance of \( R_2 = 1.44 \) (m\( \Omega \)), leakage flux that corresponds to \( L_I = 21 \) (mH), and \( L_O = 19 \) (µH), and a realistic core characterized by \( R_c = 160 \) (k\( \Omega \)) and \( L_m = 450 \) (H). The low voltage side of the transformer is at 60 (Hz), and \( V_2 = 120 \) (V), and the power there is \( P_2 = 20 \) (kW) at \( pf = 0.85 \) lagging. Calculate the voltage at the high voltage side and the efficiency of the transformer.

1. First calculate the impedances of the equivalent circuit.

\[
\begin{align*}
V_{1b} &= 4000 \text{ (V)} \\
S_{1b} &= 30 \text{ (kVA)} \\
I_{1b} &= \frac{30 \times 10^3}{4 \times 10^3} = 7.5 \text{ (A)} \\
Z_{1b} &= \frac{V_{1b}^2}{S_{1b}} = 533 \text{ (}\Omega\text{)} \\
V_{2b} &= 120 \text{ (V)} \\
S_{2b} &= S_{1b} = 30 \text{ (kVA)} \\
I_{2b} &= \frac{S_{2b}}{V_{2b}} = 250 \text{ (A)}
\end{align*}
\]
\[ Z_{2b} = \frac{V_{2b}}{I_{2b}} = 0.48 \, (\Omega) \]

2. Convert everything to per unit.

\[ R_{1,pu} = \frac{R_1}{Z_{1b}} = 0.003 \, (\text{pu}) \]
\[ R_{2,pu} = \frac{R_2}{Z_{2b}} = 0.003 \, (\text{pu}) \]
\[ R_{c,pu} = \frac{R_c}{Z_{1b}} = 300 \, (\text{pu}) \]
\[ X_{I1,pu} = \frac{2\pi \cdot 60 \cdot L_{I1}}{Z_{1b}} = 0.0149 \, (\text{pu}) \]
\[ X_{I2,pu} = \frac{2\pi \cdot 60 \cdot L_{I2}}{Z_{2b}} = 0.0149 \, (\text{pu}) \]
\[ X_{m,pu} = \frac{2\pi \cdot 60 \cdot L_m}{Z_{1b}} = 318 \, (\text{pu}) \]
\[ V_{2,pu} = \frac{V_2}{V_{2b}} = 1 \, (\text{pu}) \]
\[ P_{2,pu} = \frac{P_2}{S_{2b}} = 0.6667 \, (\text{pu}) \]

3. Solve in the per unit system.

\[ \hat{I}_{2,pu} = \frac{P_{2,pu}}{(V_{2,pu} \cdot pf)} \arccos(pf) = 0.666 - j0.413 \, (\text{pu}) \]
\[ \hat{V}_{1,pu} = \hat{V}_{2,pu} + \hat{I}_{2,pu} (R_{eq} + jX_{eq}) = 1.0163 + j0.01736 \, (\text{pu}) \]
\[ \hat{I}_{m,pu} = \frac{\hat{V}_{1,pu}}{R_{c,pu}} + \frac{\hat{V}_{1,pu}}{jX_{m,pu}} = 0.0034 - j0.0031 \, (\text{pu}) \]
\[ \hat{I}_{1,pu} = \hat{I}_{m,pu} + \hat{I}_{2,pu} = 0.6701 - j0.4163 \, (\text{pu}) \]
\[ P_{eq} = \hat{I}_{2,pu}^2 R_{eq,pu} = 0.00369 \, (\text{pu}) \]
\[ P_{c,pu} = \frac{V_{1,pu}^2}{R_{c,pu}} = 0.00344 \, (\text{pu}) \]
\[ \eta = \frac{P_{2,pu}}{(P_{2,pu} + P_{\text{loss},pu})} = \frac{0.6667}{0.6667 + (0.0037 + 0.0034)} = 0.9894 \]
4. Convert back to SI units. The efficiency is unitless and thus stays the same. Converting \( \hat{V}_1 \) gives

\[
\hat{V}_1 = V_{1b} \cdot \hat{V}_{1,pu} = 4000(1.0163 + j0.01736) = 4065.8 + j69.46 = 4065.8/0.979^\circ \text{ (V)}
\]

3.5 Testing Transformers

Purchased transformers often give information on the frequency, winding ratio, power, and voltage ratings, but not the impedances. These impedances are important in calculating the voltage regulation, efficiency, etc. Use of open circuit and short circuit tests we will determine \( R_{eq}, L_{eq}, R_c, \) and \( L_m. \)

**Open Circuit Test**

Leaving one side of the transformer open circuited, while the other has the rated voltage \( \{V_{in-oc} = 1(\text{pu})\} \) applied to it, we measure the current and power. The current that flows into the transformer is mostly determined by the impedances \( X_m \) and \( R_c, \) and it is much lower than the rated current for the transformer. It is often the case that the rated voltage for the low voltage side (low tension side) of the transform since for the above example it would be easier applying 120 (V) instead of 4000 (V). Since the units we use indicate if we are using the per unit system, the following calculations will drop the subscript \( \text{pu}. \) Using the following equivalent circuit with the primary open circuited, and \( V_2 = V_{oc} = 120 \text{ (V)} \) applied to the secondary.

![Figure 17. Open circuit testing of a 4000/120 rated transformer. With 120 (V) applied to the low voltage side \( \{V_2 = 120 \text{ (V)}\}, \) the primary voltage for this open circuit test is 4000 (V).](image)

For this open circuit test, the following can be compared to the measured values of current and power as:

\[
P_{oc} = \frac{V_{oc}^2}{R_c} = \frac{1}{R_c} \text{ (pu)} \quad (3.22)
\]

\[
\hat{i}_{oc} = \frac{V_{oc}}{R_c} \frac{V_{oc} }{jX_m} \quad (3.23)
\]
Then from the open circuit test with measurements of the current and the power, we determine $R_c$ and $X_m$.

**Short Circuit Test**

For the above open circuit test, the low voltage side of the transformer was chosen since it is easier to apply the lower voltage during that test. Similarly, the rated current is lower on the high voltage side of the transformer. With the short circuit test, the rated current is commonly applied to the high voltage side of the transformer (since with a short circuited secondary, the applied voltage required to reach the rated current is relatively low). With the rated current applied to the high voltage side, we measure the voltage, $V_{sc}$ which is $V_1$ for this example, and the power, $P_{sc}$.

$$P_{sc} = I_{sc}^2 R_{eq} = 1 \cdot R_{eq} \text{ (pu)}$$

$$\hat{V}_{sc} = \hat{I}_{sc} (R_{eq} + jX_{eq})$$

$$V_{sc} = 1 \sqrt{R_{eq}^2 + X_{eq}^2} \text{ (pu)}$$

![Diagram](image_url)

Figure 18. Short circuit testing of a 4000/120, 30 (kVA) rated transformer. This corresponds to the rated current on the high voltage side of 7.5 (A).

Then from the short circuit test with measurements of the current and the power, we determine $R_{eq} = R_1 + R_2'$ and $X_m = 2\pi \cdot 60 (L_{l1} + L'_{l2})$.

**Example**

A 60Hz transformer is rated 30 (kVA), 4000(V)/120(V). The open circuit test, with the high voltage side open, gives $P_{oc} = 100$ (W), $I_{oc} = 1.1455$ (A). The short circuit test, measured with the low voltage side shorted, gives $P_{sc} = 180$ (W), $V_{sc} = 129.79$ (V). Determine the equivalent circuit for this transformer by using the per unit system.
1. Bases:

\[ V_{1b} = 4000 \text{ (V)} \]
\[ S_{1b} = 30 \text{ (kVA)} \]
\[ I_{1b} = \frac{30 \times 10^3}{4 \times 10^3} = 7.5 \text{ (A)} \]
\[ Z_{1b} = \frac{V_{1b}^2}{S_{1b}} = 533 \text{ (Ω)} \]
\[ V_{2b} = 120 \text{ (V)} \]
\[ S_{2b} = S_{1b} = 30 \text{ (kVA)} \]
\[ I_{2b} = \frac{S_{1b}}{V_{2b}} = \frac{30 \times 10^3}{120} = 250 \text{ (A)} \]
\[ Z_{2b} = \frac{V_{2b}}{I_{2b}} = \frac{120}{250} = 0.48 \text{ (Ω)} \]

2. Convert to (pu):

\[ P_{sc,pu} = \frac{180}{30 \times 10^3} = 0.006 \text{ (pu)} \]
\[ V_{sc,pu} = \frac{129.79}{4 \times 10^3} = 0.0324 \text{ (pu)} \]
\[ P_{oc,pu} = \frac{100}{30 \times 10^3} = 0.00333 \text{ (pu)} \]
\[ I_{oc,pu} = \frac{1.1455}{250} = 0.00458 \text{ (pu)} \]

3. Calculate the components of the equivalent circuit {dropping the (pu) subscripts}

\[ P_{sc} = I_{sc}^2 R_{eq} \quad \text{or} \quad R_{eq} = \frac{P_{sc}}{I_{sc}^2} = 1 \cdot P_{sc} = 0.006 \text{ (pu)} \]
\[ |V_{sc}| = V_{sc} = |I_{sc} R_{eq} + j X_{eq}| = 1 \cdot \sqrt{R_{eq}^2 + X_{eq}^2} \quad \text{or} \quad X_{eq} = \sqrt{V_{sc}^2 - R_{eq}^2} = 0.0318 \text{ (pu)} \]
\[ P_{oc} = \frac{V_{oc}^2}{R_c} \quad \text{or} \quad R_c = \frac{V_{oc}^2}{P_{oc}} = 300 \text{ (pu)} \]
\[ |\hat{I}_{oc}| = I_{oc} = \sqrt{\frac{\hat{V}_{oc} + j\hat{I}_{oc} X_m}{R_c}} = \sqrt{\frac{1}{R_c^2} + \frac{1}{X_m^2}} \quad \text{or} \quad X_m = \frac{1}{\sqrt{\frac{I_{oc}^2 - 1}{R_c^2}}} = 318 \text{ (pu)} \]

With this knowledge, we can address the more common example of:

A 60Hz transformer is rated 30 (kVA), 4000(V)/120(V). The short circuit impedance is 0.0324 (pu) and the open circuit current is 0.0046 (pu). The rated core losses are 100 (W) and the rated winding losses are 180 (W). Calculate the efficiency and the necessary primary voltage when the load at the secondary is at rated voltage, 20 (kW), and at 0.8 pf leading.

Using (pu) system:

\[ Z_{sc} = 0.0324 \text{ (pu)} \]
\[ P_{sc} = I_{sc}^2 R_{eq} = 1 \cdot R_{eq} = \frac{180}{30 \times 10^3} = 0.006 \text{ (pu)} \]
\[ X_{eq} = \sqrt{Z_{sc}^2 - R_{eq}^2} = 0.017 \text{ (pu)} \]
\[ P_{oc} = \frac{1}{R_c} \quad \text{or} \quad R_c = \frac{1}{P_{oc}} = \frac{1}{\frac{100}{30 \times 10^3}} = 300 \text{ (pu)} \]
\[ I_{oc} = \sqrt{\frac{1}{R_c^2} + \frac{1}{X_m^2}} \quad \text{or} \quad X_m = \sqrt{\frac{I_{oc}^2 - 1}{R_c^2}} = 318 \text{ (pu)} \]

Now we have the equivalent circuit components of the transformer and can work on the full circuit which includes the load that has a power of 20 (kW) or:

\[ P_2 = \frac{20 \times 10^3}{30 \times 10^3} = 0.6667 \text{ (pu)} \]

but the power at the load is

\[ P_2 = V_2 I_2 \cdot pf \quad \text{or} \quad 0.6667 = 1 \cdot I_2 \cdot 0.8 \quad \text{thus} \quad I_2 = 0.8333 \text{ (pu)} \]

as a phasor:

\[ \hat{I}_2 = 0.8333/36.87^\circ = 0.6667 - j0.5 \text{ (pu)} \]
\[ \hat{V}_1 = \hat{V}_2 + \hat{I}_2 (R_{eq} + jX_{eq}) = 1.0125 + j0.008334 = 1.0125/0.472^\circ \text{ (pu)} \]
\[ V_1 = 1.0125 \text{ (pu)} \]
\[ P_{R_{eq}} = I_2^2 \cdot R_{eq} = 0.0062 \text{ (pu)} \]
\[ P_c = \frac{V_c^2}{R_c} = 0.034 \text{(pu)} \]

\[ \eta = \frac{P_2}{P_2 + P_{eq} + P_c} = 0.986 \]

Converting \( V_1 \) to SI units gives

\[ V_1 = V_{1,pu} \cdot V_{1b} = 1.0125 \cdot 4000 = 4050 \text{(V)} \]

### 3.6 Three-Phase Transformers

If we consider three-phase transformers as consisting of three identical one-phase transformers, then we have an accurate representation as far as equivalent circuits and two-port models are concerned, but it does not give insight into the magnetic circuit of the three-phase transformers.

The primaries and the secondaries of the one-phase transformers can be connected either in \( \Delta \) or in \( Y \) configurations. In either case, the rated power of the three-phase transformer is three times that of the one-phase transformers.

*For the \( \Delta \) connection:*

\[ V_{l-l} = V_{1\phi} \quad (3.28) \]

\[ I_l = \sqrt{3} I_{1\phi} \quad (3.29) \]

*For the \( Y \) connection:*

\[ V_{l-l} = \sqrt{3} V_{1\phi} \quad (3.30) \]

\[ I_l = I_{1\phi} \quad (3.31) \]

Connections to the three-phase transformer that are \( Y \) connected in the primary are shown in Figure 19.
Connections to the three-phase transformer that are Δ connected in the primary are shown in Figure 20.

Figure 20. Δ – Y and Δ – Δ connections of three-phase transformers.
3.7 Autotransformers

An autotransformer is a transformer where the two windings (of turns \( N_1 \) and \( N_2 \)) are not isolated from each other, but are connected as shown in Figure 21.

![Figure 21. An autotransformer.](image)

From this figure that the voltage ratio in an autotransformer is:

\[
\frac{V_1}{V_2} = \frac{N_1 + N_2}{N_2}
\]

(3.32)

and the current ratio is:

\[
\frac{I_2}{I_1} = \frac{N_1 + N_2}{N_2}
\]

(3.33)

An interesting note on autotransformers is that the coil of turns \( N_1 \) carries current \( I_1 \), while the coil of turns \( N_2 \) carries the (vectorial) sum of the two currents, \( \hat{I}_1 - \hat{I}_2 \). So if the voltage ratio were 1, no current would flow through the \( N_2 \) coil. This characteristic leads to a significant reduction in the size of the autotransformer compared to a similarly rated transformer, especially if the primary and secondary voltages are of the same order of magnitude. These savings come at a serious disadvantage of the loss of isolation between the two sides.
Chapter Notes:

- To understand the operation of transformers we have to use both the Gauss’s law for magnetism (or Biot-Savart law) and Faraday’s law.

- Most transformers operate under or near rated voltage. The voltage drop in the winding resistance and leakage reactance are usually small.

- In both transformers and in rotating machines, the net mmf of all the currents must accordingly adjust itself to create the resultant flux required by this voltage balance.

- Leakage fluxes induce voltage in the windings that are accounted for in the equivalent circuit as leakage reactance (elements $L_{11}$ and $L_{12}$). The leakage-flux paths are dominated by paths through air, and are thus almost linearly proportional to the currents producing them. The leakage reactances therefore are often assumed to be constant (independent of the degree of saturation of the core material).

- The open- and short-circuit tests provide the parameters for the equivalent circuit of the transformer.

- Three-phase transformers can be considered to be made of three single-phase transformers for the purpose of this course. The main issue is then to calculate the ratings, voltages, and currents of each.

- Autotransformers are used mostly to vary the voltage a little. It is seldom that an autotransformer will have a voltage ratio greater than two.