Autocorrelation on Symbol Dynamics for a Chaotic Dry-Friction Oscillator

B. F. Feeny†
F. C. Moon‡

Cornell University, Ithaca, New York 14853

Abstract

An autocorrelation function based on symbol dynamics is applied to a chaotic dry-friction oscillator to estimate the largest Lyapunov exponent. The friction problem is well suited for symbol dynamics since two distinct states of motion can be identified: sticking and slipping. In addition, the dynamics of the oscillator can be reduced to a non-invertible one-dimensional map, which has been studied in terms of binary symbol sequences. The study is done for an experimental oscillator and for a numerical model. The numerical result is compared to the Lyapunov exponent estimated from the continuous flow.

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† Graduate Research Assistant, Theoretical and Applied Mechanics
‡ Professor and Director, Sibley School of Mechanical and Aerospace Engineers
Introduction

Experimentalists often wish to quantitatively characterize chaotic motion by calculating the Lyapunov spectrum. However, it is computationally difficult to estimate Lyapunov exponents directly from time series data, especially in the presence of experimental noise. Singh and Joseph [1] proposed the use of symbol dynamics to obtain an autocorrelation and an estimate of the largest Lyapunov exponent. Thus, a binary sequence of yes-no information can be used to quantitatively characterize the dynamics. In this paper we report the successful use of this technique on the chaotic dynamics of a dry-friction oscillator.


In this letter, we present results for a one-degree-of-freedom oscillator with dry friction dependent on both displacement and velocity. This can occur, for example, if displacement induces elastic deformation which, in turn, causes changes in the normal load at the friction surface. The nondimensionalized equation of motion is

$$\ddot{x} + 2\zeta \dot{x} + x + n(x)f(\dot{x}) = a \cos(\Omega t),$$

(1)

where \(f(\dot{x})\) represents the friction coefficient and \(n(x)\) represents the normal load. We let
the normal load vary linearly with displacement, and, for physical consistency, we restrict
the normal load to be nonnegative, so that \( n(x) = 1 + kx \) for \( x > -1/k \), and \( n(x) = 0 \)
for \( x < -1/k \). Often, the friction is modeled by a Coulomb law, which includes a static
coefficient of friction \( \mu_s \), and a dynamic coefficient of friction \( \mu_d \). If \( \mu_s = \mu_d = \mu \), the
friction law may be written as

\[
f(\dot{x}) = \mu \text{sign}(\dot{x}).
\]  \hspace{1cm} (2)

One might try to approximate a Coulomb law with a continuous function such as

\[
f(\dot{x}) = (\mu_d + (\mu_s - \mu_d) \text{sech}(\beta \dot{x})) \tanh(\alpha \dot{x}).
\]  \hspace{1cm} (3)

The \( \tanh \) term represents the jump from positive friction to negative friction, and ap-
proaches a discontinuity as \( \alpha \to \infty \). The \( \text{sech} \) term represents the transition from \( \mu_s \) to
\( \mu_d \).

The numerical solution for the continuous case is performed using a standard fifth
order Runge-Kutta code with stepsize adjuster. The numerical solution for the discontin-
uous case (not studied in this note) requires a special algorithm which follows that of ref.
[3]. A three-dimensional representation of a numerical solution of the continuous equation
of motion is shown in Figure 1. In this plot we can see the stick-slip motion (described
in [2] and [3]). The sticking region is plotted with small dots, and the slipping motion
is plotted with large dots. Also, on the upper right portion of the portrait, we can see
trajectories from the inside of the attractor stretching above the sticking region and folding
back onto the outside of the attractor. This stretching and folding is typical of chaotic
one-dimensional maps and two-dimensional horseshoe maps. A similarly shaped attractor
was found from the experimental dry-friction oscillator described below.
Description of Experiment

The experiment consisted of a mass attached to the end of a cantilevered elastic beam. A diagram is shown in Figure 2. The mass had titanium plates on both sides, providing surfaces for sliding friction. Spring-loaded titanium pads rested against the titanium plates. The titanium plates were not parallel in the direction of sliding, thus a displacement of the mass caused a change in the force on the spring-loaded pads. Hence a change in displacement caused a change in normal load and friction. The elastic beam, mass and pressure pads were fixed to a common frame which was excited harmonically by an electromagnetic shaker. Strain gages attached to the elastic beam were used to sense the displacement of the mass relative to the oscillating frame.

Discussion of Results

An experimental Poincaré section is shown in Figure 3a. It represents a slice of the three-dimensional phase portrait of Figure 1 parallel to the $x$-$z$ axes, perpendicular to the $\Omega t$ axis [7]. The standard autocorrelation of the experimental strain signal is shown in Figure 4. It consists of a rapid decay into a small oscillation, suggesting that the signal is uncorrelated, although the influence of the harmonic driver is present. The autocorrelation of a periodic signal would be periodic, with no component of decay.

Notice that the Poincaré map appears to be confined to a one-dimensional object embedded in two-dimensional phase space. By defining a coordinate $s$ along the Poincaré map as shown in Figure 3a, we can obtain a return map as shown in Figure 3b. The return map (Figure 3b) resembles a tent map. The tent map is well known to be chaotic, and the
dynamics of similar maps have been studied in terms of binary symbol sequences[8].

The Poincaré section in Figure 3a is located near the axes labeled in Figure 1. Note that it cuts through the sticking region. The sharp, horizontal part of the Poincaré plot is in the sticking region, and the fuzzy part is in slipping motion. This suggests a natural use of symbol dynamics for the motion, namely sticking or slipping. By considering whether the motion is sticking or slipping at each pass through the Poincaré section, we can construct a binary symbol sequence.

Singh and Joseph [1] have proposed a technique of extracting quantitative information from a binary symbol sequence. First it is necessary to represent the symbol sequence \( u(k) \) as a string of 1's and -1's. These values are chosen so that the expected mean of a random sequence of equally likely symbols is zero. As the trajectory passes through the Poincaré section for the \( k \)th time, if it is not sticking, we set \( u(k) = 1 \). If it is sticking, we set \( u(k) = -1 \). An autocorrelation on such a symbol sequence is defined as

\[
    r(n) = \frac{1}{N} \sum_{k=1}^{N} u(k + n)u(k), \quad n = 0, 1, 2, \ldots, \quad N \gg n. \tag{4}
\]

If the sequence is chaotic, the autocorrelation should have the property \( r(n) \to 0 \) as \( n \to \infty \).

If the sequence becomes uncorrelated, an estimate of the largest Lyapunov exponent can be obtained using the binary autocorrelation function. The largest Lyapunov exponent can be defined as

\[
    \lambda = \frac{1}{N} \sum_{i=1}^{N} \log_2 \frac{d(n)}{d_0(n-1)}, \tag{5}
\]

where \( d_0(n-1) \) is the starting distance between two trajectories, and \( d(n) \) is the distance between them after one iteration. Since the binary sequence is uncorrelated, we can estimate \( d_0(n-1) \) as the expected value of the distance \( \bar{d}_0(n-1) \) between two randomly
chosen points in the same symbol region. In our example, we measure the distance using coordinate $s$ on the Poincaré plot. Two points chosen from the sticking region have an expected distance $\bar{d}_o(n-1) = 1/3$. Two points from the nonsticking region also have an expected distance $\bar{d}_o(n-1) = 1/3$. If $u(n-1)$ and $u(n)$ are in the same region, their iterates will either stay in that region, be in different regions, or both be in the other region. One defines[1]

$$\alpha = \log_2 \frac{\bar{d}(n)}{\bar{d}_o(n-1)},$$

(6)

where $\bar{d}(n)$ is the expected distance of two points, each chosen from separate regions. For our problem, $\bar{d}(n) = 1$ and $\alpha = \log_2 3$. Replacing $d_o(n-1)$ and $d(n)$ in (5) with their expected values defines the macroscopic Lyapunov exponent, $\lambda_m$, which is rewritten via a derivation in ref. [1] as

$$\lambda = \frac{\alpha}{2}(1 - r(1)^2),$$

(7)

Application of equations (4), (6) and (7) to a symbol sequence derived from the tent map yields a rapidly decaying autocorrelation and a Lyapunov exponent $\lambda_t = 0.787516$ for a string of 100000 symbols, and an exponent of $\lambda_t = 0.787705$ for a string of 2048 symbols, compared to its exact value, calculated using $\log_2$, $\lambda_{te} = 1$. Application to the logistic map yields a rapidly decaying autocorrelation and a Lyapunov exponent of $\lambda_l = 0.791578$ for 100000 symbols, and $\lambda_l = 0.791116$ for 2048 symbols, compared to its exact value of $\lambda_{te} = 1$.

The binary autocorrelation function for an experimental sequence of length 2048 was obtained using (4) as shown in Figure 5a. Applying equations (6) and (7), the resulting Lyapunov exponent is $\lambda_{exp} = 0.79055$. Using equations (4), (6) and (7) on numerical
smooth-law data (2048 symbols) yields the autocorrelation in Figure 5b, and a Lyapunov
exponent of $\lambda_{s_1} = 0.79219$. The largest Lyapunov exponent of the flow can be estimated
numerically[7], and can be related to that of the Poincaré map via $\lambda_{flow} = \frac{\lambda}{T}$, where $T$ is
the driving period. This calculation of exponent for the Poincaré map from the equations
of motion gave $\lambda_{s_2} = 0.77$.

Conclusions

Binary symbol dynamics were used to describe the stick-slip motion of a dry-friction
oscillator. Sticking and slipping motion were used as the states in the binary sequence. It
was shown that the dynamics of the dry-friction oscillator is reducible to a one-dimensional
map, well suited for symbol dynamics. The proposed method in ref. [1] produced a binary
autocorrelation function which was used to estimate the order of magnitude of the largest
Lyapunov exponent. This estimate was done for experimental and numerical data. The
implication is that for limited information, i.e. a series of 'yes' and 'no' information,
quantitative information of the dynamics can be easily obtained.

References

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Figure Captions

Figure 1. Three-dimensional phase portrait of the numerical solution using a continuous friction law with parameters $\mu_s = 1$, $\mu_d = 0.7$, $\zeta = 0.015$, $\alpha = 50$, $\beta = 5$, $\Omega = 1.3$, and $a = 1.45$.

Figure 2. Schematic of experiment.

Figure 3. (a) Experimental Poincaré section, (b) Return map of Poincaré section.

Figure 4. Standard autocorrelation function for experimental time series.

Figure 5. Binary autocorrelation function for (a) symbol sequence from experimental data, (b) symbol sequence from numerical simulation.
Feeny and Moon, Fig 5b