1 Basic Vector Calculus and Algebra

1.1 Notations

Cartesian Coordinates \( \hat{x}, \hat{y}, \hat{z} \)
Cylindrical Coordinates \( \hat{\rho}, \hat{\phi}, \hat{z} \)
Spherical Coordinates \( \hat{r}, \hat{\theta}, \hat{\phi} \)

Position Vector \( \mathbf{r} = x \hat{x} + y \hat{y} + z \hat{z} \)

NOTE: All vector will be denoted using either \textbf{boldface} or a bar, eg., \textbf{a} or \( \bar{a} \). All dyads (a term that shall be explained shortly) will be identified using upper case letters as in \( \mathcal{A} \) or using a \textbf{bar} over an uppercase letter \( \bar{A} \).

There are two kinds of notation that is commonly used. The direct form that was mentioned earlier, and the index notation. The index notation is sometimes easier to understand and manipulate. For instance, a vector can be denoted either as \( \mathbf{a} \) or equivalently \( a_i \) for \( i = 1, 2, 3 \) (we are assuming three dimensional space). One way of interpreting this is to assume that \( i = 1 \) represents \( \hat{x} \), \( i = 2 \) represents \( \hat{y} \) and \( i = 3 \) represents \( \hat{z} \), i.e., each number stands for a component.

While the index notation is mathematically useful, it should be remembered that each index corresponds to a mapping of a vector on to a directional basis that are used, and hence, are coordinate dependent. The direct form is coordinate independent and represents the quantity as is.

1.2 Basic Vector Identities

\[
\begin{align*}
\mathbf{a} \cdot \mathbf{a} &= |\mathbf{a}|^2 \\
\mathbf{a} \cdot \mathbf{a}^* &= |\mathbf{a}|^2 \\
\mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} \\
\mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} \\
\mathbf{a} \times \mathbf{b} &= -\mathbf{b} \times \mathbf{a} \\
(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} &= \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \\
(\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \\
\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} &= \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} \\
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= ?
\end{align*}
\]

Using index notation, the dot product may be written as \( \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i \). Frequently, the sum is implicitly assumed, and the dot product is written as \( a_i b_i \). The summation convention implies that the repeated indices are summed over. In this case, \( a_i b_i = \sum_{i=1}^{3} a_i b_i \). The same holds for more complicated dot products that we shall come across later.

Also, a dot product implies contraction. This is best understood using index notation:

\[
\begin{align*}
a_i b_i &= \text{constant} \implies \text{scalar} \quad (1) \\
A_{ij} b_j &= c_i \implies \text{vector} \quad (2) \\
A_{ijk} b_k &= B_{ij} \implies \text{Dyad} \quad (3)
\end{align*}
\]

In other words, taking a dot product with an \( n^{th} \) order tensor gives you an \( (n-1)^{th} \) order tensor.

It is also apparent that a dot product may be written as \( a_i b_j \delta_{ij} \) where \( \delta_{ij} = 1 \) for \( i = j \) and 0 otherwise. It is also a good exercise to prove that \( \mathbf{a} \times \mathbf{b} = \hat{x} a_j b_k \epsilon_{ijk} \), where \( \hat{x} \) is the \( i^{th} \) unit vector, and

\[
\epsilon_{ijk} = \begin{cases} 
1 & \text{if } i, j, k \text{ are cyclic} \\
-1 & \text{if } i, j, k \text{ are not cyclic} \\
0 & \text{if } i, j, k \text{ are repeated}
\end{cases}
\]
1.3 \nabla \text{ operator}

1.3.1 Orthogonal Coordinate System

Consider the position vector \( \mathbf{r} \), and an orthogonal coordinate system where the unit vectors are given by \( \hat{x}_1 \), \( \hat{x}_2 \) and \( \hat{x}_3 \). We want to transform a vector in the \( xyz \) coordinate system to \( x_1x_2x_3 \). To do so, we will assume that \( x = f_1(x_1, x_2, x_3), y = f_2(x_1, x_2, x_3), \) and \( z = f_3(x_1, x_2, x_3) \).

\[
\mathbf{r} = x \hat{x} + y \hat{y} + z \hat{z} = f_1 \hat{x} + f_2 \hat{y} + f_3 \hat{z}
\]

From the above we can compute

\[
d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x_1} dx_1 + \frac{\partial \mathbf{r}}{\partial x_2} dx_2 + \frac{\partial \mathbf{r}}{\partial x_3} dx_3
\]

(5)

The vector \( \frac{\partial \mathbf{r}}{\partial x_1} \) is tangential to the \( x_1 \) coordinate curve. Thus, in terms of unit vectors, once can write eqn. ?? as

\[
d\mathbf{r} = h_1 dx_1 \hat{x}_1 + h_2 dx_2 \hat{x}_2 + +h_3 dx_3 \hat{x}_3
\]

(6)

where \( h_1 \hat{x}_1 = \frac{\partial \mathbf{r}}{\partial x_1} \) and likewise for the other coordinates. The arc length and the volume are

\[
ds = d\mathbf{r} \cdot d\mathbf{r} = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2
\]

(7a)

\[
dV = |(h_1 dx_1 \hat{x}_1) \cdot (h_2 dx_2 \hat{x}_2) \times (h_3 dx_3 \hat{x}_3)| = h_1 h_2 h_3 dx_1 dx_2 dx_3
\]

(7b)

\[
\begin{vmatrix}
\frac{\partial x_1}{\partial y_1} & \frac{\partial y_1}{\partial y_2} & \frac{\partial y_1}{\partial y_3} \\
\frac{\partial x_2}{\partial y_1} & \frac{\partial y_2}{\partial y_2} & \frac{\partial y_2}{\partial y_3} \\
\frac{\partial x_3}{\partial y_1} & \frac{\partial y_3}{\partial y_2} & \frac{\partial y_3}{\partial y_3}
\end{vmatrix}
\]

\[
dx_1 dx_2 dx_3
\]

(7c)

1.3.2 Generic div, curl and Laplacian

Using these, the following hold true:

\[
\nabla \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial x_1} \hat{x}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial x_2} \hat{x}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial x_3} \hat{x}_3
\]

(8a)

\[
\nabla \cdot \mathbf{u} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} (h_2 h_3 u_1) + \frac{\partial}{\partial x_2} (h_1 h_3 u_2) + \frac{\partial}{\partial x_3} (h_1 h_2 u_3) \right]
\]

(8b)

\[
\nabla \times \mathbf{u} = \begin{vmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
\frac{h_1 \hat{x}_1}{h_1} & \frac{h_2 \hat{x}_2}{h_2} & \frac{h_3 \hat{x}_3}{h_3} \\
\frac{h_1 u_1}{h_1} & \frac{h_2 u_2}{h_2} & \frac{h_3 u_3}{h_3}
\end{vmatrix}
\]

(8c)

\[
\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial x_1} \right) + \left( \frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial x_2} \right) + \left( \frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial x_3} \right) \right]
\]

(8d)

1.3.3 Coordinate Specific

**Cartesian:** The volume element is given by \( dx \, dy \, dz \implies h_1 = 1; \, h_2 = 1; \, h_3 = 1 \). This is the trivial case.

**Cylindrical:** The volume element is given by \( r \, dr \, \rho \, d\phi \, dz \implies h_1 = 1; \, h_2 = \rho; \, h_3 = 1 \).

**Spherical:** The volume element is given by \( dr \, r \, d\theta \, r \sin \theta \, d\phi \implies h_1 = 1; \, h_2 = r; \, h_3 = r \sin \theta \)
1.4 Stokes and Gauss’ Laws

In addition to the above theorem, integral theorems in the form of Gauss and Stokes theorem occur very often in electromagnetics. Their direct mapping into KCL and KVL equations will be taught in the next class. Given a vector $\mathbf{E}(\mathbf{r})$ that is defined $\forall \mathbf{r} \in V$ and that $V$ is bounded by a surface $S$, we can write

$$\int_S ds \, \hat{n} \cdot \mathbf{E}(\mathbf{r}) = \int_v dv \nabla \cdot \mathbf{E}(\mathbf{r}) \quad (9)$$

$$\oint \mathbf{r} \cdot \mathbf{E}(\mathbf{r}) = \int_s ds \, \hat{n} \cdot (\nabla \times \mathbf{E}(\mathbf{r})) \quad (10)$$

This is only a partial list of identities. The burning question is how does all of this help in understanding Electromagnetics. You will the use of this in the next class and as we progress through the rest of this course.