Nonlinear Systems and Control
Lecture # 5
Limit Cycles
Oscillation: A system oscillates when it has a nontrivial periodic solution

\[ x(t + T) = x(t), \quad \forall \ t \geq 0 \]

Linear (Harmonic) Oscillator:

\[
\dot{z} = \begin{bmatrix}
0 & -\beta \\
\beta & 0 \\
\end{bmatrix} z
\]

\[ z_1(t) = r_0 \cos(\beta t + \theta_0), \quad z_2(t) = r_0 \sin(\beta t + \theta_0) \]

\[ r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \quad \theta_0 = \tan^{-1} \left[ \frac{z_2(0)}{z_1(0)} \right] \]
The linear oscillation is not practical because

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation).

- The amplitude of oscillation depends on the initial conditions.

The same problems exist with oscillation of nonlinear systems due to a center equilibrium point (e.g., pendulum without friction).
Limit Cycles:

Example: Negative Resistance Oscillator

\[ i = h(v) \]
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \varepsilon h'(x_1) x_2 
\end{align*}
\]

There is a unique equilibrium point at the origin

\[
A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}
\]

\[
\lambda^2 + \varepsilon h'(0) \lambda + 1 = 0
\]

\[h'(0) < 0 \Rightarrow \text{Unstable Focus or Unstable Node}\]
Energy Analysis:

\[ E = \frac{1}{2} C v_C^2 + \frac{1}{2} L \dot{i}_L^2 \]

\[ v_C = x_1 \quad \text{and} \quad \dot{i}_L = -h(x_1) - \frac{1}{\varepsilon} x_2 \]

\[ E = \frac{1}{2} C \left\{ x_1^2 + [\varepsilon h(x_1) + x_2]^2 \right\} \]

\[ \dot{E} = C \left\{ x_1 \dot{x}_1 + [\varepsilon h(x_1) + x_2] [\varepsilon h'(x_1) \dot{x}_1 + \dot{x}_2] \right\} \]

\[ = C \left\{ x_1 x_2 + [\varepsilon h(x_1) + x_2] [\varepsilon h'(x_1) x_2 - x_1 - \varepsilon h'(x_1) x_2] \right\} \]

\[ = C [x_1 x_2 - \varepsilon x_1 h(x_1) - x_1 x_2] \]

\[ = -\varepsilon C x_1 h(x_1) \]
\[ \dot{E} = -\epsilon C x_1 h(x_1) \]
Example: Van der Pol Oscillator

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2
\end{align*}
\]

\[\varepsilon = 0.2\]

\[\varepsilon = 1\]
\[ \dot{z}_1 = \frac{1}{\varepsilon}z_2 \]

\[ \dot{z}_2 = -\varepsilon(z_1 - z_2 + \frac{1}{3}z_2^3) \]

\[ \varepsilon = 5 \]
Stable Limit Cycle

Unstable Limit Cycle