Nonlinear Systems and Control
Lecture # 25

Stabilization

Basic Concepts & Linearization
We want to stabilize the system

\[ \dot{x} = f(x, u) \]

at the equilibrium point \( x = x_{ss} \)

**Steady-State Problem:** Find steady-state control \( u_{ss} \) s.t.

\[ 0 = f(x_{ss}, u_{ss}) \]

\[ x_\delta = x - x_{ss}, \quad u_\delta = u - u_{ss} \]

\[ \dot{x}_\delta = f(x_{ss} + x_\delta, u_{ss} + u_\delta) \overset{\text{def}}{=} f_\delta(x_\delta, u_\delta) \]

\[ f_\delta(0, 0) = 0 \]

\[ u_\delta = \gamma(x_\delta) \implies u = u_{ss} + \gamma(x - x_{ss}) \]
State Feedback Stabilization: Given

\[ \dot{x} = f(x, u) \quad [f(0, 0) = 0] \]

find

\[ u = \gamma(x) \quad [\gamma(0) = 0] \]

s.t. the origin is an asymptotically stable equilibrium point of

\[ \dot{x} = f(x, \gamma(x)) \]

\( f \) and \( \gamma \) are locally Lipschitz functions
Linear Systems

\[ \dot{x} = Ax + Bu \]

\((A, B)\) is stabilizable (controllable or every uncontrollable eigenvalue has a negative real part)

Find \(K\) such that \((A - BK)\) is Hurwitz

\[ u = -Kx \]

Typical methods:

- Eigenvalue Placement
- Eigenvalue-Eigenvector Placement
- LQR
Linearization

\[ \dot{x} = f(x, u) \]

\( f(0, 0) = 0 \) and \( f \) is continuously differentiable in a domain \( D_x \times D_u \) that contains the origin \( (x = 0, u = 0) \)

\( (D_x \subset \mathbb{R}^n, D_u \subset \mathbb{R}^p) \)

\[ \dot{x} = Ax + Bu \]

\[ A = \left. \frac{\partial f}{\partial x} (x, u) \right|_{x=0,u=0} ; \quad B = \left. \frac{\partial f}{\partial u} (x, u) \right|_{x=0,u=0} \]

Assume \((A, B)\) is stabilizable. Design a matrix \( K \) such that \((A - BK)\) is Hurwitz

\[ u = -Kx \]
Closed-loop system:

\[ \dot{x} = f(x, -Kx) \]

\[
\dot{x} = \left[ \frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx) (-K) \right]_{x=0} x
\]

\[ = (A - BK)x \]

Since \((A - BK)\) is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system
Example (Pendulum Equation):

\[ \ddot{\theta} = -a \sin \theta - b \dot{\theta} + cT \]

Stabilize the pendulum at \( \theta = \delta \)

\[ 0 = -a \sin \delta + cT_{ss} \]

\[ x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{ss} \]

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin \delta] - bx_2 + cu
\end{align*} \]

\[ A = \begin{bmatrix}
0 & 1 \\
-a \cos(x_1 + \delta) & -b
\end{bmatrix} \bigg|_{x_1=0} = \begin{bmatrix}
0 & 1 \\
-a \cos \delta & -b
\end{bmatrix} \]
\[ A = \begin{bmatrix} 0 & 1 \\ -a \cos \delta & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix} \]

\[ K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \]

\[ A - BK = \begin{bmatrix} 0 & 1 \\ -(a \cos \delta + ck_1) & -(b + ck_2) \end{bmatrix} \]

\[ k_1 > -\frac{a \cos \delta}{c}, \quad k_2 > -\frac{b}{c} \]

\[ T = \frac{a \sin \delta}{c} - Kx = \frac{a \sin \delta}{c} - k_1(\theta - \delta) - k_2\dot{\theta} \]
Notions of Stabilization

\[ \dot{x} = f(x, u), \quad u = \gamma(x) \]

Local Stabilization: The origin of \( \dot{x} = f(x, \gamma(x)) \) is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of \( \dot{x} = f(x, \gamma(x)) \) is asymptotically stable and a given region \( G \) is a subset of the region of attraction (for all \( x(0) \in G, \lim_{t \to \infty} x(t) = 0 \)) (e.g., \( G \subset \Omega_c = \{ V(x) \leq c \} \) where \( \Omega_c \) is an estimate of the region of attraction)

Global Stabilization: The origin of \( \dot{x} = f(x, \gamma(x)) \) is globally asymptotically stable
Semiglobal Stabilization: The origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable and $\gamma(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \gamma_p(x)$ is dependent on a parameter $p$ such that for any compact set $G$, $p$ can be chosen to ensure that $G$ is a subset of the region of attraction)

What is the difference between global stabilization and semiglobal stabilization?
Example

\[ \dot{x} = x^2 + u \]

Linearization:

\[ \dot{x} = u, \quad u = -kx, \quad k > 0 \]

Closed-loop system:

\[ \dot{x} = -kx + x^2 \]

Linearization of the closed-loop system yields \( \dot{x} = -kx \).
Thus, \( u = -kx \) achieves local stabilization

The region of attraction is \( \{ x < k \} \). Thus, for any set \( \{ x \leq a \} \) with \( a < k \), the control \( u = -kx \) achieves regional stabilization
The control $u = -kx$ does not achieve global stabilization.

But it achieves **semiglobal stabilization** because any compact set $\{|x| \leq r\}$ can be included in the region of attraction by choosing $k > r$.

The control

$$u = -x^2 - kx$$

achieves **global stabilization** because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable.
Practical Stabilization

\[ \dot{x} = f(x, u) + g(x, u, t) \]

\[ f(0, 0) = 0, \quad g(0, 0, t) \neq 0 \]

\[ \|g(x, u, t)\| \leq \delta, \quad \forall x \in D_x, u \in D_u, t \geq 0 \]

There is no control \( u = \gamma(x) \), with \( \gamma(0) = 0 \), that can make the origin of

\[ \dot{x} = f(x, \gamma(x)) + g(x, \gamma(x), t) \]

uniformly asymptotically stable because the origin is not an equilibrium point.
**Definition:** The system

\[
\dot{x} = f(x, u) + g(x, u, t)
\]

is **practically stabilizable** if for any \( \beta > 0 \) there is a control law \( u = \gamma(x) \) such that the solutions of

\[
\dot{x} = f(x, \gamma(x)) + g(x, \gamma(x), t)
\]

are uniformly ultimately bounded by \( \beta \); i.e.,

\[
\|x(t)\| \leq \beta, \quad \forall t \geq T
\]

Typically, \( u = \gamma_p(x) \) is dependent on a parameter \( p \) such that for any \( \beta > 0 \), \( p \) can be chosen to ensure that \( \beta \) is an ultimate bound.
With practical stabilization, we may have
- local practical stabilization
- regional practical stabilization
- global practical stabilization, or
- semiglobal practical stabilization

depending on the region of initial states
Example

\[ \dot{x} = x^2 + u + d(t), \quad |d(t)| \leq \delta, \quad \forall \ t \geq 0 \]

\[ u = -kx, \ k > 0, \ \Rightarrow \dot{x} = x^2 - kx + d(t) \]

\[ V = \frac{1}{2}x^2 \ \Rightarrow \dot{V} = x^3 - kx^2 + xd(t) \]

\[ \dot{V} \leq -\frac{k}{3}x^2 - x^2 \left( \frac{k}{3} - |x| \right) - |x| \left( \frac{k}{3} |x| - \delta \right) \]

\[ \dot{V} \leq -\frac{k}{3}x^2, \quad \text{for} \ \mu := \frac{3\delta}{k} \leq |x| \leq \frac{k}{3} \]

Take \[ \frac{3\delta}{k} = \alpha_1^{-1}(\alpha_2(\mu)) \leq \beta \ \Leftrightarrow \ k \geq \frac{3\delta}{\beta} \]

By choosing \( k \) large enough we can achieve semiglobal practical stabilization
\[ \dot{x} = x^2 + u + d(t) \]

\[ u = -x^2 - kx, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx + d(t) \]

\[ V = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V} = -kx^2 + xd(t) \]

\[ \dot{V} \leq -\frac{k}{2}x^2 - |x| \left( \frac{k}{2} |x| - \delta \right) \]

\[ \dot{V} \leq -\frac{k}{2}x^2, \quad \text{for } |x| \geq \frac{2\delta}{k} =: \mu \]

\[ \Rightarrow \beta \geq \alpha_1^{-1}(\alpha_2(\mu)) = \mu \]

By choosing \( k \) large enough we can achieve global practical stabilization