Nonlinear Systems and Control
Lecture # 15
Positive Real Transfer Functions
&
Connection with Lyapunov Stability
Definition: A $p \times p$ proper rational transfer function matrix $G(s)$ is positive real if

- poles of all elements of $G(s)$ are in $\text{Re}[s] \leq 0$
- for all real $\omega$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega} (s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is called strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$
Scalar Case \((p = 1)\):

\[
G(j\omega) + G^T(-j\omega) = 2\text{Re}[G(j\omega)]
\]

\(\text{Re}[G(j\omega)]\) is an even function of \(\omega\).

The second condition of the definition reduces to

\[
\text{Re}[G(j\omega)] \geq 0, \ \forall \ \omega \in [0, \infty)
\]

which holds when the Nyquist plot of \(G(j\omega)\) lies in the
closed right-half complex plane

This is true only if the relative degree of the transfer function
is zero or one

Note: for \(G(s) = \frac{n(s)}{d(s)}\), the relative degree is \(\text{deg}d-\text{deg}n\).
\[ G(j\omega) = \frac{1}{j\omega + 1} \]
\[ G(j\omega) = \frac{1}{(j\omega)^2 + j\omega + 1} \]
Lemma: Suppose $\det [G(s) + G^T(-s)]$ is not identically zero. Then, $G(s)$ is strictly positive real if and only if

1. $G(s)$ is Hurwitz
2. $G(j\omega) + G^T(-j\omega) > 0$, $\forall \omega \in \mathbb{R}$
3. $G(\infty) + G^T(\infty) > 0$ or it is positive semidefinite and
   \[
   \lim_{\omega \to \infty} \omega^2 M^T [G(j\omega) + G^T(-j\omega)] M > 0
   \]

   for any $p \times (p - q)$ full-rank matrix $M$ such that
   \[
   M^T [G(\infty) + G^T(\infty)] M = 0 \in \mathbb{R}^{(p-q)\times(p-q)}
   \]

   $q = \text{rank}[G(\infty) + G^T(\infty)]$
If $G(\infty) + G^T(\infty)$ is singular, the third condition ensures that $G(j\omega) + G^T(-j\omega)$ has

- $q$ singular values with
  \[
  \lim_{\omega \to \infty} \sigma_i(\omega) > 0
  \]

- $(p - q)$ singular values with
  \[
  \lim_{\omega \to \infty} \sigma_i(\omega) = 0, \quad \lim_{\omega \to \infty} \omega^2 \sigma_i(\omega) > 0
  \]
Scalar Case \((p = 1)\): \(G(s)\) is strictly positive real if and only if

- \(G(s)\) is Hurwitz
- \(\text{Re}[G(j\omega)] > 0, \ \forall \omega \in [0, \infty)\)
- \(G(\infty) > 0\) or \(G(\infty) = 0\) and

\[
\lim_{\omega \to \infty} \omega^2 \text{Re}[G(j\omega)] > 0
\]
Example:

\[ G(s) = \frac{1}{s} \]

has a simple pole at \( s = 0 \) whose residue is 1

\[ \text{Re}[G(j\omega)] = \text{Re} \left[ \frac{1}{j\omega} \right] = 0, \ \forall \omega \neq 0 \]

Hence, \( G \) is positive real. It is not strictly positive real since

\[ \frac{1}{(s - \varepsilon)} \]

has a pole in \( \text{Re}[s] > 0 \) for any \( \varepsilon > 0 \)
Example:

\[ G(s) = \frac{1}{s + a}, \quad a > 0, \quad \text{is Hurwitz} \]

\[ \Re\{G(j\omega)\} = \frac{a}{\omega^2 + a^2} > 0, \quad \forall \ \omega \in [0, \infty) \]

\[ \lim_{\omega \to \infty} \omega^2 \Re\{G(j\omega)\} = \lim_{\omega \to \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0 \ \Rightarrow \ \text{G is SPR} \]

Example:

\[ G(s) = \frac{1}{s^2 + s + 1}, \quad \Re\{G(j\omega)\} = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2} \]

G is not PR
Example:

\[ G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ -\frac{1}{s+2} & \frac{2}{s+1} \end{bmatrix} \text{ is Hurwitz} \]

\[
G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & -\frac{2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} > 0, \ \forall \ \omega \in \mathbb{R}
\]

\[
G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
\lim_{\omega \to \infty} \omega^2 M^T [G(j\omega) + G^T(-j\omega)] M = 4 \quad \Rightarrow \quad G \text{ is SPR}
\]
Positive Real Lemma: Let

\[ G(s) = C(sI - A)^{-1}B + D \]

where \((A, B)\) is controllable and \((A, C)\) is observable. \(G(s)\) is positive real if and only if there exist matrices \(P = P^T > 0, L, \) and \(W\) such that

\[
PA + A^TP = -L^TL
\]
\[
PB = C^T - L^TW
\]
\[
W^TW = D + D^T
\]
Kalman–Yakubovich–Popov Lemma: Let

\[ G(s) = C(sI - A)^{-1}B + D \]

where \((A, B)\) is controllable and \((A, C)\) is observable. \(G(s)\) is strictly positive real if and only if there exist matrices \(P = P^T > 0\), \(L\), and \(W\), and a positive constant \(\varepsilon\) such that

\[
PA + A^TP = -L^TL - \varepsilon P \\
PB = C^T - L^TW \\
W^TW = D + D^T
\]
**Lemma:** The linear time-invariant minimal realization

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

with

\[
G(s) = C(sI - A)^{-1}B + D
\]

is

- passive if \( G(s) \) is positive real
- strictly passive if \( G(s) \) is strictly positive real

**Proof:** Apply the PR and KYP Lemmas, respectively, and use \( V(x) = \frac{1}{2}x^TPx \) as the storage function
\[
u^T y - \frac{\partial V}{\partial x} f(x, u) = u^T y - \frac{\partial V}{\partial x} (Ax + Bu)
\]
\[
= u^T (Cx + Du) - x^T P (Ax + Bu)
\]
\[
= u^T Cx + \frac{1}{2} u^T (D + D^T) u
\]
\[
- \frac{1}{2} x^T (PA + A^T P)x - x^T PBu
\]
\[
= u^T (B^T P + W^T L)x + \frac{1}{2} u^T W^T W u
\]
\[
+ \frac{1}{2} x^T L^T Lx + \frac{1}{2} \varepsilon x^T P x - x^T PBu
\]
\[
= \frac{1}{2} (Lx + Wu)^T (Lx + Wu) + \frac{1}{2} \varepsilon x^T P x \geq \frac{1}{2} \varepsilon x^T P x
\]

In the case of the PR Lemma, \(\varepsilon = 0\), and we conclude that the system is passive; in the case of the KYP Lemma, \(\varepsilon > 0\), and we conclude that the system is strictly passive.
Connection with Lyapunov Stability

**Lemma:** If the system

\[
\dot{x} = f(x, u), \quad y = h(x, u)
\]

is passive with a positive definite storage function \(V(x)\), then the origin of \(\dot{x} = f(x, 0)\) is stable

**Proof:**

\[
u^T y \geq \frac{\partial V}{\partial x} f(x, u) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, 0) \leq 0
\]
Lemma: If the system

\[ \dot{x} = f(x, u), \quad y = h(x, u) \]

is strictly passive, then the origin of \( \dot{x} = f(x, 0) \) is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Proof: The storage function \( V(x) \) is positive definite

\[ u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \psi(x) \Rightarrow \frac{\partial V}{\partial x} f(x, 0) \leq -\psi(x) \]

Why is \( V(x) \) positive definite?
Let \( \phi(t; x) \) be the solution of \( \dot{z} = f(z, 0), \ z(0) = x \)
\[
\dot{V} \leq -\psi(x)
\]

\[
V(\phi(\tau, x)) - V(x) \leq - \int_{0}^{\tau} \psi(\phi(t; x)) \, dt, \quad \forall \tau \in [0, \delta]
\]

\[
V(\phi(\tau, x)) \geq 0 \Rightarrow V(x) \geq \int_{0}^{\tau} \psi(\phi(t; x)) \, dt
\]

\[
V(\bar{x}) = 0 \Rightarrow \int_{0}^{\tau} \psi(\phi(t; \bar{x})) \, dt = 0, \quad \forall \tau \in [0, \delta]
\]

\[
\Rightarrow \psi(\phi(t; \bar{x})) \equiv 0 \Rightarrow \phi(t; \bar{x}) \equiv 0 \Rightarrow \bar{x} = 0
\]
Definition: The system

\[ \dot{x} = f(x, u), \quad y = h(x, u) \]

is zero-state observable if no solution of \( \dot{x} = f(x, 0) \) can stay identically in \( S = \{ h(x, 0) = 0 \} \), other than the zero solution \( x(t) \equiv 0 \)

Linear Systems

\[ \dot{x} = Ax, \quad y = Cx \]

Observability of \((A, C)\) is equivalent to

\[ y(t) = Ce^{At}x(0) \equiv 0 \iff x(0) = 0 \iff x(t) \equiv 0 \]
If \((A, C)\) is observable

\[ y(t) = Ce^{At}x(0) \equiv 0 \Leftrightarrow x(0) = 0 \]

**Proof:** \((\Leftarrow)\) trivial

\((\Rightarrow)\) Suppose not, i.e., \(y(t) = Ce^{At}x(0) \equiv 0 \Rightarrow x(0) \neq 0\)

Cayley Hamilton

\[
\left[ \sum_{k=0}^{n-1} \alpha_k(t)CA^k \right] x(0) \equiv 0
\]

\[
\Leftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(0) \equiv 0
\]
Lemma: If the system

\[
\dot{x} = f(x,u), \quad y = h(x,u)
\]

is output strictly passive and zero-state observable, then the origin of \( \dot{x} = f(x,0) \) is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Proof: The storage function \( V(x) \) is positive definite

\[
u^T y \geq \frac{\partial V}{\partial x} f(x,u) + y^T \rho(y) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x,0) \leq -y^T \rho(y)\]

\[
\dot{V}(x(t)) \equiv 0 \quad \Rightarrow \quad y(t) \equiv 0 \quad \Rightarrow \quad x(t) \equiv 0
\]

Apply the invariance principle.
Example

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0 \]

\[ V(x) = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2 \]

\[ \dot{V} = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu \]

The system is output strictly passive

\[ y(t) \equiv 0 \iff x_2(t) \equiv 0 \Rightarrow ax_1^3(t) \equiv 0 \Rightarrow x_1(t) \equiv 0 \]

The system is zero-state observable. \( V \) is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable