Nominal System:

\[ \dot{x} = f(x), \quad f(0) = 0 \]

Perturbed System:

\[ \dot{x} = f(x) + g(t, x), \quad g(t, 0) = 0 \]

Case 1: The origin of the nominal system is exponentially stable

\[ c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \]

\[ \frac{\partial V}{\partial x} f(x) \leq -c_3 \|x\|^2 \]

\[ \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\| \]
Use $V(x)$ as a Lyapunov function candidate for the perturbed system

$$\dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x)$$

Assume that

$$\|g(t, x)\| \leq \gamma \|x\|, \quad \gamma \geq 0$$

$$\dot{V}(t, x) \leq -c_3 \|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \|g(t, x)\|$$

$$\leq -c_3 \|x\|^2 + c_4 \gamma \|x\|^2$$
\[ \gamma < \frac{c_3}{c_4} \]

\[ \dot{V}(t, x) \leq -(c_3 - \gamma c_4)\|x\|^2 \]

The origin is an exponentially stable equilibrium point of the perturbed system.
Example

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3, \quad \beta \geq 0
\end{align*} \]

\[ \dot{x} = Ax + g(x) \]

\[ A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix} \]

The eigenvalues of \( A \) are \(-1 \pm j\sqrt{3}\)

\[ PA + A^T P = -I \implies P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix} \]
For the quadratic Lyapunov function $V(x) = x^T P x$, 

\[
\lambda_{\text{min}}(P) \|x\|_2^2 \leq V(x) \leq \lambda_{\text{max}}(P) \|x\|_2^2 \\
=: c_1 = c_2
\]

\[
\mathcal{L}_{Ax} V = \frac{\partial V}{\partial x} Ax = -x^T Q x \leq -\lambda_{\text{min}}(Q) \|x\|_2^2 \\
=: c_3
\]

\[
\left\| \frac{\partial V}{\partial x} \right\|_2 = \|2x^T P\| \leq 2\|P\|_2 \|x\|_2 = 2\lambda_{\text{max}}(P) \|x\|_2 \\
=: c_4
\]
\[ V(x) = x^T P x, \quad \frac{\partial V}{\partial x} A x = -x^T x \]

\[ c_3 = 1, \quad c_4 = 2 \quad \|P\| = 2\lambda_{\text{max}}(P) = 2 \times 1.513 = 3.026 \]

\[ \|g(x)\| = \beta |x_2|^3 \leq \beta k_2^2 |x_2| \leq \beta k_2^2 \|x\|, \quad \forall |x_2| \leq k_2 \]

\(g(x)\) satisfies the bound \(\|g(x)\| \leq \gamma \|x\|\) over compact sets of \(x\). Consider the compact set

\[ \Omega_c = \{V(x) \leq c\} = \{x^T P x \leq c\}, \quad c > 0 \]

\[ k_2 = \max_{x^T P x \leq c} |x_2| = \max_{x^T P x \leq c} |[0 1]x| \]
Fact:
\[
\max_{x^TPx \leq c} \|Lx\| = \sqrt{c} \|LP^{-1/2}\|
\]

Proof
\[
x^TPx \leq c \iff \frac{1}{c} x^TPx \leq 1 \iff \frac{1}{c} x^TP^{1/2} P^{1/2} x \leq 1
\]
\[
y = \frac{1}{\sqrt{c}} P^{1/2} x
\]
\[
\max_{x^TPx \leq c} \|Lx\| = \max_{y^Ty \leq 1} \|L\sqrt{c} P^{-1/2} y\| = \sqrt{c} \|LP^{-1/2}\|
\]
\[ k_2 = \max_{x^T P x \leq c} |[0 \ 1] x| = \sqrt{c} \|[0 \ 1] P^{-1/2}\| = 1.8194 \sqrt{c} \]

\[ \|g(x)\| \leq \beta c (1.8194)^2 \|x\|, \quad \forall x \in \Omega_c \]

\[ \|g(x)\| \leq \gamma \|x\|, \quad \forall x \in \Omega_c, \quad \gamma = \beta c (1.8194)^2 \]

\[ \gamma < \frac{c_3}{c_4} \iff \beta < \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c} \]

\[ \beta < \frac{0.1}{c} \Rightarrow \dot{V}(x) \leq -(1 - 10\beta c) \|x\|^2 \]

Hence, the origin is exponentially stable and \( \Omega_c \) is an estimate of the region of attraction.
Alternative Bound on $\beta$

$$\dot{V}(x) = -\|x\|^2 + 2x^T Pg(x)$$
$$= -\|x\|^2 + \frac{1}{8}\beta x_2^3 ([2\ 5] x)$$
$$\leq -\|x\|^2 + \sqrt[8]{29}\beta x_2^2 \|x\|^2$$

Over $\Omega_c$, $x_2^2 \leq (1.8194)^2 c$

$$\dot{V}(x) \leq -\left(1 - \frac{\sqrt[8]{29}}{8}\beta (1.8194)^2 c\right) \|x\|^2$$
$$= -\left(1 - \frac{\beta c}{0.448}\right) \|x\|^2$$

If $\beta < 0.448/c$, the origin will be exponentially stable and $\Omega_c$ will be an estimate of the region of attraction
Remark: The inequality $\beta < \frac{0.448}{c}$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on $\beta$. The smaller the upper bound on $\beta$, the larger the estimate of RA.
Case 2: The origin of the nominal system is asymptotically stable

\[
\dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x) \leq -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t, x) \right\|
\]

Under what condition will the following inequality hold?

\[
\left\| \frac{\partial V}{\partial x} g(t, x) \right\| < W_3(x)
\]

Special Case: Quadratic-Type Lyapunov function

\[
\frac{\partial V}{\partial x} f(x) \leq -c_3 \phi^2(x), \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \phi(x)
\]
\[ \phi(x) : \mathbb{R}^n \to \mathbb{R} \text{ is positive definite and continuous} \]

\[ \dot{V}(t, x) \leq -c_3 \phi^2(x) + c_4 \phi(x) \|g(t, x)\| \]

If \[ \|g(t, x)\| \leq \gamma \phi(x) \], with \( \gamma < \frac{c_3}{c_4} \)

\[ \dot{V}(t, x) \leq -(c_3 - c_4 \gamma) \phi^2(x) \]
Example

\[ \dot{x} = -x^3 + g(t, x) \]

\( V(x) = x^4 \) is a quadratic-type Lyapunov function for the nominal system \( \dot{x} = -x^3 \)

\[ \frac{\partial V}{\partial x}(-x^3) = -4x^6, \quad \left| \frac{\partial V}{\partial x} \right| = 4|x|^3 \]

\( \phi(x) = |x|^3, \quad c_3 = 4, \quad c_4 = 4 \)

Suppose \( |g(t, x)| \leq \gamma|x|^3, \quad \forall \ x, \) with \( \gamma < 1 \)

\[ \dot{V}(t, x) \leq -4(1 - \gamma)\phi^2(x) \]

Hence, the origin is a globally uniformly asymptotically stable
Remark: A nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds.

Example

\[ \dot{x} = -x^3 + \gamma x \]

The origin is unstable for any \( \gamma > 0 \) (can be easily seen via linearization)