Nonlinear Systems and Control
Lecture # 11
Exponential Stability
&
Region of Attraction
Exponential Stability:
The origin of $\dot{x} = f(x)$ is exponentially stable if and only if the linearization of $f(x)$ at the origin is Hurwitz

**Theorem:** Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable function such that

$$k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a$$

$$\dot{V}(x) \leq -k_3 \|x\|^a$$

for all $x \in D$, where $k_1$, $k_2$, $k_3$, and $a$ are positive constants. Then, the origin is an exponentially stable equilibrium point of $\dot{x} = f(x)$. If the assumptions hold globally, the origin will be globally exponentially stable.
Proof: Choose $c > 0$ small enough that

$$\{k_1 \|x\|^a \leq c\} \subset D$$

$$V(x) \leq c \Rightarrow k_1 \|x\|^a \leq c$$

$$\Omega_c = \{V(x) \leq c\} \subset \{k_1 \|x\|^a \leq c\} \subset D$$

$\Omega_c$ is compact and positively invariant; $\forall x(0) \in \Omega_c$

$$\dot{V} \leq -k_3 \|x\|^a \leq -\frac{k_3}{k_2} V$$

$$\frac{dV}{V} \leq - \frac{k_3}{k_2} dt$$

$$V(x(t)) \leq V(x(0)) e^{-(k_3/k_2)t}$$
\[
\| x(t) \| \leq \left[ \frac{V(x(t))}{k_1} \right]^{1/a} \leq \left[ \frac{V(x(0)) e^{-(k_3/k_2)t}}{k_1} \right]^{1/a} \leq \left[ \frac{k_2 \| x(0) \|^a e^{-(k_3/k_2)t}}{k_1} \right]^{1/a} = \left( \frac{k_2}{k_1} \right)^{1/a} e^{-\gamma t} \| x(0) \|, \quad \gamma = \frac{k_3}{(k_2 a)}
\]
Example

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1) - x_2 
\end{align*} \]

\[ c_1 y^2 \leq y h(y) \leq c_2 y^2, \quad \forall \ y, \ c_1 > 0, \ c_2 > 0 \]

\[ V(x) = \frac{1}{2} x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2 \int_0^{x_1} h(y) \, dy \]

\[ c_1 x_1^2 \leq 2 \int_0^{x_1} h(y) \, dy \leq c_2 x_1^2 \]

\[ \dot{V} = [x_1 + x_2 + 2h(x_1)]x_2 + [x_1 + 2x_2][-h(x_1) - x_2] \]

\[ = -x_1 h(x_1) - x_2^2 \leq -c_1 x_1^2 - x_2^2 \]

The origin is globally exponentially stable
Region of Attraction

**Lemma:** If $x = 0$ is an asymptotically stable equilibrium point for $\dot{x} = f(x)$, then its region of attraction $R_A$ is an open, connected, invariant set. Moreover, the boundary of $R_A$ is formed by trajectories
Example

\[ \begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 + (x_1^2 - 1)x_2
\end{align*} \]
Example

\[ \begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2
\end{align*} \]
Estimates of the Region of Attraction: Find a subset of the region of attraction

Warning: Let $D$ be a domain with $0 \in D$ such that for all $x \in D$, $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite

Is $D$ a subset of the region of attraction? NO

Why?
Example: Reconsider

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \frac{1}{3} x_1^3 - x_2
\end{align*} \]

\[ V(x) = \frac{1}{2} x^T \begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix} x + 2 \int_0^{x_1} (y - \frac{1}{3} y^3) \, dy \\
&= \frac{3}{2} x_1^2 - \frac{1}{6} x_1^4 + x_1 x_2 + x_2^2
\]

\[ \dot{V}(x) = -x_1^2 (1 - \frac{1}{3} x_1^2) - x_2^2 \]

\[ D = \{ -\sqrt{3} < x_1 < \sqrt{3} \} \]

Is \( D \) a subset of the region of attraction?
The simplest estimate is the bounded component of \( \{ V(x) < c \} \), where \( c = \min_{x \in \partial D} V(x) \)

For \( V(x) = x^T P x \), where \( P = P^T > 0 \), the minimum of \( V(x) \) over \( \partial D \) is given by

For \( D = \{ \| x \| < r \} \), \( \quad \min_{\| x \| = r} x^T P x = \lambda_{\min}(P) r^2 \)

For \( D = \{ |b^T x| < r \} \), \( \quad \min_{|b^T x| = r} x^T P x = \frac{r^2}{b^T P^{-1} b} \)

For \( D = \{ |b_i^T x| < r_i, \ i = 1, \ldots, p \} \),

Take \( c = \min_{1 \leq i \leq p} \frac{r_i^2}{b_i^T P^{-1} b_i} \leq \min_{x \in \partial D} x^T P x \)
Example (Revisited)

\[ \begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 + (x_1^2 - 1)x_2
\end{align*} \]

\[ V(x) = 1.5x_1^2 - x_1x_2 + x_2^2 \]

\[ \dot{V}(x) = -(x_1^2 + x_2^2) - (x_1^3x_2 - 2x_1^2x_2^2) \]

\[ \dot{V}(x) < 0 \text{ for } 0 < \|x\|^2 < \frac{2}{\sqrt{5}} \overset{\text{def}}{=} r^2 \]

Take \( c = \lambda_{\text{min}}(P)r^2 = 0.691 \times \frac{2}{\sqrt{5}} = 0.618 \)

\{V(x) < c\} is an estimate of the region of attraction
\[ x_1 = \rho \cos \theta, \quad x_2 = \rho \sin \theta \]

\[
\dot{V} = -\rho^2 + \rho^4 \cos^2 \theta \sin \theta (2 \sin \theta - \cos \theta)
\leq -\rho^2 + \rho^4 |\cos^2 \theta \sin \theta| \cdot |2 \sin \theta - \cos \theta|
\leq -\rho^2 + \rho^4 \times 0.3849 \times 2.2361
\leq -\rho^2 + 0.861\rho^4 < 0, \text{ for } \rho^2 < \frac{1}{0.861}
\]

Take \( c = \lambda_{\text{min}}(P)r^2 = \frac{0.691}{0.861} = 0.803 \)

Alternatively, choose \( c \) as the largest constant such that \( \{x^TPx < c\} \) is a subset of \( \{\dot{V}(x) < 0\} \)
(a) Contours of $\dot{V}(x) = 0$ (dashed) $V(x) = 0.8$ (dash-dot), $V(x) = 2.25$ (solid)
(b) comparison of the region of attraction with its estimate
If $D$ is a domain where $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite (or $\dot{V}(x)$ is negative semidefinite and no solution can stay identically in the set $\dot{V}(x) = 0$ other than $x = 0$), then according to LaSalle’s theorem any compact positively invariant subset of $D$ is a subset of the region of attraction.

**Example**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4(x_1 + x_2) - h(x_1 + x_2)
\end{align*}
\]

$h(0) = 0; \ uh(u) \geq 0, \ \forall \ |u| \leq 1$
\[
V(x) = x^T P x = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2
\]

\[
\dot{V}(x) = (4x_1 + 2x_2) \dot{x}_1 + 2(x_1 + x_2) \dot{x}_2
\]
\[
= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2)
\]
\[
\leq -2x_1^2 - 6(x_1 + x_2)^2, \quad \forall |x_1 + x_2| \leq 1
\]
\[
= -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x
\]

\[\dot{V}(x)\] is negative definite in \(|x_1 + x_2| \leq 1\)

\[b^T = [1 \ 1], \quad c = \min_{|x_1 + x_2| = 1} x^T P x = \frac{1}{b^T P^{-1} b} = 1\]
\[ \sigma = x_1 + x_2 \]

\[ \frac{d}{dt} \sigma^2 = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \leq 2\sigma x_2 - 8\sigma^2, \quad \forall |\sigma| \leq 1 \]

On \( \sigma = 1 \), \[ \frac{d}{dt} \sigma^2 \leq 2x_2 - 8 \leq 0, \quad \forall x_2 \leq 4 \]

On \( \sigma = -1 \), \[ \frac{d}{dt} \sigma^2 \leq -2x_2 - 8 \leq 0, \quad \forall x_2 \geq -4 \]

\[ c_1 = V(x)|_{x_1=-3,x_2=4} = 10, \quad c_2 = V(x)|_{x_1=3,x_2=-4} = 10 \]

\[ \Gamma = \{ V(x) \leq 10 \text{ and } |x_1 + x_2| \leq 1 \} \]

is a subset of the region of attraction
\[ V(x) = 10 \]
\[ V(x) = 1 \]

(-3, 4)

(3, -4)