Nonlinear Systems and Control
Lecture # 10
The Invariance Principle
Example: Pendulum equation with friction

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a \sin x_1 - bx_2
\end{align*}
\]

\[
V(x) = a(1 - \cos x_1) + \frac{1}{2} x_2^2
\]

\[
\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2 \dot{x}_2 = -bx_2^2
\]

The origin is stable. \(\dot{V}(x)\) is not negative definite because \(\dot{V}(x) = 0\) for \(x_2 = 0\) irrespective of the value of \(x_1\)

However, near the origin \(x_1 \neq 0\), the solution cannot stay identically in the set \(\{x_2 = 0\}\)
Definitions: Let $x(t)$ be a solution of $\dot{x} = f(x)$

A point $p$ is said to be a **positive limit point** of $x(t)$ if there is a sequence $\{t_n\}$, with $\lim_{n \to \infty} t_n = \infty$, such that $x(t_n) \to p$ as $n \to \infty$

The set of all positive limit points of $x(t)$ is called the **positive limit set** of $x(t)$; denoted by $L^+$

If $x(t)$ approaches an asymptotically stable equilibrium point $\bar{x}$, then $\bar{x}$ is the **positive limit point** of $x(t)$ and $L^+ = \bar{x}$

A stable limit cycle is the **positive limit set** of every solution starting sufficiently near the limit cycle.
A set $M$ is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in \mathbb{R}$$

Examples:

- Equilibrium points
- Limit Cycles

A set $M$ is a *positively invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

Example:

The set $\Omega_c = \{ V(x) \leq c \}$ with $\dot{V}(x) \leq 0$ in $\Omega_c$
The distance from a point $p$ to a set $M$ is defined by

$$\text{dist}(p, M) = \inf_{x \in M} \| p - x \|$$

$x(t)$ approaches a set $M$ as $t$ approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\text{dist}(x(t), M) < \varepsilon, \; \forall \; t > T$$

**Example:** every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \to \infty$

Notice, however, that $x(t)$ does not converge to any specific point on the limit cycle
Lemma: If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to $D$ for $t \geq 0$, then its positive limit set $L^+$ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches $L^+$ as $t \to \infty$

LaSalle’s theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over $D$ such that $\dot{V}(x) \leq 0$ in $\Omega$. Let $E$ be the set of all points in $\Omega$ where $\dot{V}(x) = 0$, and $M$ be the largest invariant set in $E$. Then every solution starting in $\Omega$ approaches $M$ as $t \to \infty$
Note:

- Here “largest” invariant set is understood in the sense of set theory, i.e., $M$ is the union of all invariant sets (such as equilibrium points or limit cycles) within $E$.

- If once $\dot{V}(x) = 0$, then $\dot{V}(x) \equiv 0$ for all future time, then $M = E$.

- Here $V(x)$ does not have to be positive definite.
Proof:

\[ \dot{V}(x) \leq 0 \text{ in } \Omega \text{(compact)} \implies V(x(t)) \text{ is a decreasing} \]

\[ V(x) \text{ is continuous in } \Omega \implies V(x) \geq b = \min_{x \in \Omega} V(x) \]

\[ \implies \lim_{t \to \infty} V(x(t)) = a \]

\[ x(t) \in \Omega \implies x(t) \text{ is bounded} \implies L^+ \text{ exists (Lemma)} \]

Moreover, \( L^+ \subset \Omega \) and \( x(t) \) approaches \( L^+ \) as \( t \to \infty \)

For any \( p \in L^+ \), there is \( \{t_n\} \) with \( \lim_{n \to \infty} t_n = \infty \) such that \( x(t_n) \to p \) as \( n \to \infty \)

\[ V(x) \text{ is continuous} \implies V(p) = \lim_{n \to \infty} V(x(t_n)) = a \]
Why?

$V(x)$ is continuous  $\Rightarrow$  $V(p) = \lim_{n \to \infty} V(x(t_n)) = a.$

$\forall \delta, \exists N, \text{ s.t. } \|x(t_n) - p\| < \delta, \forall n \geq N,$

$\forall \epsilon, \exists \delta, \text{ s.t. } \|x - p\| < \delta \Rightarrow \|V(x) - V(p)\| < \epsilon,$

$\Rightarrow \forall \epsilon, \exists N, \text{ s.t. } \|V(x(t_n)) - V(p)\| < \epsilon, \forall n \geq N$

$V(x) = a \text{ on } L^+ \text{ and } L^+ \text{ invariant } \Rightarrow \dot{V}(x) = 0, \forall x \in L^+$

$L^+ \subset M \subset E \subset \Omega$

$x(t) \text{ approaches } L^+ \Rightarrow x(t) \text{ approaches } M \text{ (as } t \to \infty)
Theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over $D$ such that $\dot{V}(x) \leq 0$ in $D$. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in $S$, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable.

- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction.

- Furthermore, if $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable.
Example:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h_1(x_1) - h_2(x_2)
\end{align*}
\]

\[
h_i(0) = 0, \quad yh_i(y) > 0, \quad \text{for } 0 < |y| < a
\]

\[
V(x) = \int_0^{x_1} h_1(y) \, dy + \frac{1}{2}x_2^2
\]

\[
D = \{ -a < x_1 < a, \quad -a < x_2 < a \}
\]

\[
\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \leq 0
\]

\[
\dot{V}(x) = 0 \Rightarrow x_2 h_2(x_2) = 0 \Rightarrow x_2 = 0
\]

\[
S = \{ x \in D \mid x_2 = 0 \}
\]
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2)
\]
\[
x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0
\]

The only solution that can stay identically in \(S\) is \(x(t) \equiv 0\)

Thus, the origin is asymptotically stable

Suppose \(a = \infty\) and \(\int_0^y h_1(z) \, dz \to \infty\) as \(|y| \to \infty\)

Then, \(D = \mathbb{R}^2\) and \(V(x) = \int_0^{x_1} h_1(y) \, dy + \frac{1}{2}x_2^2\) is radially unbounded. \(S = \{x \in \mathbb{R}^2 \mid x_2 = 0\}\) and the only solution that can stay identically in \(S\) is \(x(t) \equiv 0\)

The origin is globally asymptotically stable
Example: $m$-link Robot Manipulator

Two-link Robot Manipulator
\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u \]

- \( q \) is an \( m \)-dimensional vector of joint positions
- \( u \) is an \( m \)-dimensional control (torque) inputs

\[ M = M^T > 0 \] is the inertia matrix

\( C(q, \dot{q})\dot{q} \) accounts for centrifugal and Coriolis forces

\[
(\dot{M} - 2C)^T = -(\dot{M} - 2C)
\]

- \( D\dot{q} \) accounts for viscous damping; \( D = D^T \geq 0 \)

- \( g(q) \) accounts for gravity forces; \( g(q) = [\partial P(q)/\partial q]^T \)

- \( P(q) \) is the total potential energy of the links due to gravity
Investigate the use of the (PD plus gravity compensation) control law

\[ u = g(q) - K_p(q - q^*) - K_d \dot{q} \]

to stabilize the robot at a desired position \( q^* \), where \( K_p \) and \( K_d \) are symmetric positive definite matrices

\[ e = q - q^*, \quad \dot{e} = \dot{q} \]

\[ M \ddot{e} = M \ddot{q} \]
\[ = -C \dot{q} - D \dot{q} - g(q) + u \]
\[ = -C \dot{q} - D \dot{q} - K_p(q - q^*) - K_d \dot{q} \]
\[ = -C \dot{e} - D \dot{e} - K_p e - K_d \dot{e} \]
\[ M \ddot{e} = -C \dot{e} - D \dot{e} - K_p e - K_d \dot{e} \]

\[ V = \frac{1}{2} \dot{e}^T M(q) \dot{e} + \frac{1}{2} e^T K_p e \]

\[ \dot{V} = \dot{e}^T M \ddot{e} + \frac{1}{2} \dot{e}^T \dot{M} \dot{e} + e^T K_p \dot{e} \]

\[ = -\dot{e}^T C \dot{e} - \dot{e}^T D \dot{e} - \dot{e}^T K_p e - \dot{e}^T K_d \dot{e} \]

\[ + \frac{1}{2} \dot{e}^T \dot{M} \dot{e} + e^T K_p \dot{e} \]

\[ = \frac{1}{2} \dot{e}^T (\dot{M} - 2C) \dot{e} - \dot{e}^T (K_d + D) \dot{e} \]

\[ = -\dot{e}^T (K_d + D) \dot{e} \leq 0 \]
\[(K_d + D) \text{ is positive definite}\]

\[\dot{V} = -\dot{e}^T (K_d + D) \dot{e} = 0 \Rightarrow \dot{e} = 0\]

\[M\ddot{e} = -C \dot{e} - D \dot{e} - K_p e - K_d \dot{e}\]

\[\dot{e}(t) \equiv 0 \Rightarrow \ddot{e}(t) \equiv 0 \Rightarrow K_p e(t) \equiv 0 \Rightarrow e(t) \equiv 0\]

By LaSalle’s theorem the origin \((e = 0, \dot{e} = 0)\) is globally asymptotically stable