**Recursive, Infinite Impulse Response (IIR) Digital Filters:**

Filters defined by Laplace Domain transfer functions (analog devices) can be easily converted to Z domain transfer functions (digital, sampled data devices).

In the Laplace domain, a single time step or unit delay has the transfer function, $e^{-s\tau}$.

In the Z domain, a single time step or unit delay has the transfer function, $z^{-1}$.

Equating these two transfer functions provides a mapping between the Laplace domain and the Z domain, yields the transformation

$$z^{-1} = e^{-s\tau} \quad \Rightarrow \quad z = e^{s\tau}$$

where the time step or unit delay is given by the coefficient, $\tau$.

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**Example #1: First Order Filter Design**

Find the sampled data form of the first-order transfer function,

$$T(s) = \frac{2}{0.2s+1}$$

This transfer function has a single pole at $s = -5$ and a single zero at $s = -\infty$ that reduces all signals with frequencies above 5 radians/sec. and has a steady state gain,

$$T(\omega=0, z = 1) = 2.$$  

The frequency response of this transfer function can be found with MatLab commands

```matlab
num = [2]; den = [0.2 1]; bode(num,den);
```
To find the Z domain transfer function first specify the sampling rate. We will assume a sampling rate of 100 Hz., which is well above the “cutoff frequency”,
\[ \omega = 5 \text{ rad./sec.} = 0.796 \text{ Hz.} \]
and find the Z domain pole corresponding to the Laplace domain pole at \( s = -5 \),
\[ z = e^{s\tau} \]
where \( \tau = 1/100 = 0.01 \)
\( s = -5 \text{ rad/sec (real valued pole in this case)} \)

This yields the Z domain pole at
\[ z = e^{(-5*0.01)} = 0.951 \]
The Laplace domain zero is at “\(-\infty\)” so the Z domain zero is located at
\[ z = e^{(-\infty*0.01)} = 0.0 \]
The sampled data transfer function is now assembled with the Z domain transfer function,
\[ T(z) = (Gz)/(z-0.951) \]
where \( G \) = unknown gain constant to be specified

Note that this transfer function has z-domain poles and zeros at the positions in the z-domain corresponding to the points calculated for the transformed Laplace domain poles and zeros. The variable \( z \) represents a positive unit time step into the future that cannot
be realized. We must modify our Z domain transfer function so that it is in units of delay “z⁻¹” by multiplying top and bottom through by enough factors of z⁻¹ to yield only “delay” operators in the z-domain transfer function. In this case, we need only to multiply top and bottom by,

$$T(z) = G/(1-0.951*z^{-1})$$

where $G =$ unknown gain constant to be specified

This transfer function has the zero frequency ($z=1$) magnitude at $\omega = 0$ of

$$T(\omega=0, z = 1) = 2 = G/(1-0.951) = G/0.0488$$

This magnitude must be matched to the S-domain magnitude at zero frequency where $T(s) = T(0) = 2$ so that the Z transform $G$ parameter becomes

$$G = T(\omega=0)*0.0488 = 2*0.0488= 0.0975$$

and the final Z domain transfer function is,

$$T(z) = 0.0975/(1-0.951*z^{-1})$$

**IIR Filter Implementation:**

The derived z-domain filter must now be implemented with sampled data algebra. We start with the Z Transform,

$$Y(z)/X(z) = T(z) = G/(1-a*z^{-1})$$

and write the transfer functions in “delay operator” form on the input “X(z)” and the output “Y(z)”.

$$(1-a*z^{-1})Y(z) = G X(z)$$

Rearranging terms,

$$Y(z) - (a*z^{-1})*Y(z) = G X(z)$$

or

$$Y(z) = G X(z) + (a*z^{-1})*Y(z)$$

but $z^{-1}$ is a unit delay, so in sampled data form,

$$y(k) = G*x(k) + a*y(k-1)$$

This sampled data equation shown in block diagram form below.
Second-Order IIR Filter Design Example:

Derive the sampled data filter corresponding to the frequency response below.

For this example, pick a sampling interval corresponding to 1000 Hz. sampling frequency => \( \tau = 0.001 \) seconds

Now compute the Z-domain pole locations:

\[
\begin{align*}
    s &= 62.8 \text{ rad/sec} \implies z = e^{s \tau} = e^{-62.8 \times 0.001} = e^{-0.0628} = 0.939101 \\
    s &= 6.28 \text{ rad/sec} \implies z = e^{s \tau} = e^{-6.28 \times 0.001} = e^{-0.00628} = 0.993737
\end{align*}
\]

and compute the Z-domain zero locations:

\[
\begin{align*}
    s &= -\infty \text{ rad/sec} \implies z = e^{s \tau} = e^{-\infty} = 0.0 \\
    s &= 0.0 \text{ rad/sec} \implies z = e^{s \tau} = e^{0.00} = 1.00
\end{align*}
\]

Rewriting transfer function in Z domain,

\[
T(z) = G \left[ \frac{z(z - 1)}{(z - 0.993737)(z - 0.939101)} \right] = G \left[ \frac{z(z - 1)}{z^2 - 1.93284z + 0.933219} \right] = G \left[ \frac{1 - z^{-1}}{1 - 1.9328z^{-1} + 0.933219z^{-2}} \right]
\]

To compute gain, G, match the amplitude of T(z) at a single frequency. Here, set frequency, \( \omega \), to halfway between 6.28 rad/sec and 62.8 rad/sec on the log scale.
Amplitude matching frequency,
\[ \omega = a \log [ \log(6.28) + 0.5( \log(6.28) - \log(62.8) ) ] = 19.8569 \]

so set \( \omega = 20 \) rad/sec where amplitude of \( T(z) = 0 \) dB = 1.0

and

\[
G = \frac{1 - 1.9328z^{-1} + 0.933219z^{-2}}{1 - z^{-1}} = \frac{(z - 0.993737)(z - 0.939101)}{z(z - 1)}
\]

where: \( z = e^{j\omega \tau} = e^{j(20 \times 0.001)} = e^{j(0.020)} \)

\[
G = \frac{(z - 0.993737)(z - 0.939101)}{z(z - 1)} = \frac{(0.0209)(0.0639)}{(1)(0.020)} = 0.067
\]

our z-domain transfer function becomes:

\[
T(z) = \frac{Y(z)}{X(z)} = 0.067 \left[ \frac{1 - z^{-1}}{1 - 1.9328z^{-1} + 0.933219z^{-2}} \right]
\]

\[
(1 - 1.9328z^{-1} + 0.933219z^{-2})Y(z) = 0.067(1 - z^{-1})Y(z)
\]

\[
Y(z) = (1.9328z^{-1} - 0.933219z^{-2})Y(z) + 0.067(1 - z^{-1})Y(z)
\]

so that our filter equation becomes:

\[
y(k) = 1.9328y(k-1) - 0.933219y(k-2) + 0.067 \left[ x(k) - x(k-1) \right]
\]

This filter is shown in block diagram form below.