Linearization a la Taylor.

1. Example of a first-order system with one input.
Consider a single nonlinear state equation in standard form shown below:
\[
\dot{x}(t) = f(x, u) \tag{1.1}
\]
where \(x(t)\) is the state variable and \(u(t)\) is the input variable.

Find the equilibrium point(s) … EP(s).
Let the input be constant, \(u(t) = u_c\), and assume that \(dx/dt(t) = 0\). Then [1.2] becomes
\[
0 = f(x_{ep}, u_c) \tag{1.2}
\]
where \(x_{ep}\) is the EP value and \(uc\) is the constant input. This equation looks easy to solve and sometimes it is, but sometimes it is not. One difficulty, for example, is that there may be no solution, one solution or many solutions.

Linearize about an EP.
Assume that we have found an EP and it is at \([x_{ep}, u_c]\). Next we want to linearize the system model about that EP. To help us we call on Taylor and his famous series. The variables we will use are these:
\[
\delta(t) = x(t) - x_{ep} \quad \text{and} \quad \varepsilon(t) = u(t) - u_c.
\]
where \(\delta(t)\) is a perturbation in \(x(t)\) and \(\varepsilon(t)\) is a perturbation in \(u(t)\).

Then we can see that the following is true: \(\dot{x}(t) = \dot{\delta}(t)\).

Thus [1.1] can be written as
\[
\dot{\delta}(t) = f(x_{ep} + \delta(t), u_c + \varepsilon(t)),
\]
which Taylor assures us can be written as
\[
\dot{\delta}(t) = f(x_{ep}, u_c) + (\partial f/\partial x) \delta + (\partial f/\partial u) \varepsilon + (h.o.t.) \tag{1.3}
\]
where \((h.o.t.)\) means all terms involving any combination of \((\delta, \varepsilon)\) with order higher than one. The derivatives are evaluated at the EP. By ignoring the higher-order terms and by using [1.2] in [1.3] we get
\[
\dot{\delta}(t) = (\partial f/\partial x) \delta + (\partial f/\partial u) \varepsilon = A \delta + B \varepsilon, \tag{1.4}
\]
where we make the connection between coefficients in [1.4] as
\[
A = (\partial f/\partial x) \quad \text{and} \quad B = (\partial f/\partial u) \tag{1.5 a,b}
\]

This is our favorite equation form, is it not?
2. Example of a second-order system with two inputs.

Consider a nonlinear system with two state variables and two inputs:

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_1, x_2; u_1, u_2) \\
\dot{x}_2(t) &= f_2(x_1, x_2; u_1, u_2)
\end{align*}
\]  

[2.1]

Find the equilibrium point(s) … EP(s).

Let the input be constant, \( u(t) = u_c \), and assume that \( dx/dt(t) = 0 \). Then [2.1] becomes

\[
\begin{align*}
0 &= f_1(x_{1ep}, x_{2ep}, u_{1c}, u_{2c}) \\
0 &= f_2(x_{1ep}, x_{2ep}, u_{1c}, u_{2c})
\end{align*}
\]  

[2.2]

Solving a set of nonlinear algebraic equations is not an easy task, generally. One difficulty is that there may be no solution, one solution or many solutions. Sometimes simulation can be used to help out in looking for EPs.

Linearize about an EP.

Assume that we have found an EP and it is at \([ x_{ep}, u_c ]\), where \( x_{ep} \) and \( u_c \) are column vectors. Next we want to linearize the system model about that EP. To help us we call on Taylor and his famous series. The variables we will use are these:

\[ x(t) = x_{ep} + \delta(t) \quad \text{and} \quad u(t) = u_c + \varepsilon(t). \]

where \( \delta(t) \) is a perturbation vector in \( x(t) \) and \( \varepsilon(t) \) is a perturbation vector in \( u(t) \).

Then we can see that the following is true:

\[ \dot{x}(t) = \dot{\delta}(t). \]

Thus [2.1] can be written in compact notation as

\[ \dot{\delta}(t) = f(x_{ep} + \delta(t), u_c + \varepsilon(t)), \]

which Taylor assures us can be written as

\[ \dot{\delta}(t) = f(x_{ep}, u_c) + \left( \frac{\partial f}{\partial x} \right) \delta + \left( \frac{\partial f}{\partial u} \right) \varepsilon + (h.o.t.) \]  

[2.3]

where \( (h.o.t.) \) means all terms involving any combination of \( (\delta, \varepsilon) \) with order higher than one. The derivatives are evaluated at the EP. By ignoring the higher-order terms and by using [2.2] in [2.3] we get

\[ \dot{\delta}(t) = \left( \frac{\partial f}{\partial x} \right) \delta + \left( \frac{\partial f}{\partial u} \right) \varepsilon = A \delta + B \varepsilon, \]  

[2.4]

where we make the connection between coefficients in [2.4] as

\[
A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix} \quad \text{and} \quad
B = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2}
\end{bmatrix}
\]  

[2.5 a,b]

What could be a groovier result? (Nothing …)