1 Problem

(20 points) Consider the OP amp circuit in Figure 1. Find the transfer function from the input voltage $v_i(t)$ to the output voltage $v_o(t)$. Hint: $v_d = 0$ and $i_+ = 0$.

The circuit can be redrawn in Laplace domain as,

\[
\frac{v_i(s)}{v_d(s)} - 0 = \frac{i}{R'} \text{ where } R' = \frac{1}{R_1 + \frac{1}{sL}}
\]

\[
\frac{0 - v_o(s)}{R''} = i \text{ where } R'' = \frac{R_2 + \frac{1}{sC}}{R_1 + sL}
\]

Comparing currents,

\[
\frac{v_i(s)}{R_1 sL} = \frac{v_o(s)}{R_2 + \frac{1}{sC}}
\]
\[ \frac{V_o(s)}{V_i(s)} = -\frac{R_2sC + 1}{sC} \cdot \frac{R_1sL}{R_{isL}} \cdot \frac{1}{R_1 + sL} \]

\[ = -\frac{(R_2sC + 1)(R_1sL)}{s^2R_1LC} \]

TF \rightarrow \[ \frac{V_o(s)}{V_i(s)} = -\frac{(R_1sL)(sR_2C + 1)}{s^2R_1LC} \]
2 Problem

a. (15 points) Given the differential equation

\[ \dot{x} = -3x^3 + 3u, \]

linearize the equation about \( x_0 = 1 \). Write the linearized equation in the new coordinates such as \( \delta x \) and \( \delta u \).

b. (5 points) Find the transfer function from the input \( \delta u(t) \) to the output \( \delta x(t) \), i.e.,

\[ T_{\delta u \rightarrow \delta x}(s) = \frac{L(\delta x(t))}{L(\delta u(t))} = ? \]

\[ \dot{x} = -3x^3 + 3u, \]

For equilibrium, \( \dot{x} = 0 \), \( 2x_0 = 1 \) we get

\[ 0 = -3x(1)^3 + 3u_0 \]

or, \( u_0 = 1 \).

New co-ordinates, \( \delta x = x - x_0 \rightarrow (x-1) \)

\( \delta u = u - u_0 \rightarrow (u-1) \)

Linearizing around \((x_0, u_0)\),

\[ \delta \dot{x} = \frac{\partial^2 f}{\partial x \partial u} \bigg|_{x=x_0, u=u_0} \delta x + \frac{\partial f}{\partial u} \bigg|_{x=x_0, u=u_0} \delta u \]

\( \theta_1 \), \( \delta \dot{x} = -9(1)^2 \delta x + 3 \delta u \)

\( \theta_2 \), \( \delta \dot{x} = -9 \delta x + 3 \delta u \)

\( \theta_3 \), \( \delta \dot{x} + 9 \delta x - 3 \delta u = 0 \)

\[ \dot{\delta x} + 9 \delta x - 3 \delta u = 0 \]

\[ -\ddot{\omega} \]
Taking Laplace function of linearized equation, we get,

\[ 8d[\delta(x)] + 9d[\delta(x)] - 3d[\delta(u)] = 0 \]

Or,

\[ (8+9)d[\delta(x)] = 3d[\delta(u)] \]

\[ \therefore \frac{d[\delta(x)]}{d[\delta(u)]} = \frac{3}{8+9} \]

Ans. (b)
3 Problem

![Block Diagram]

Figure 2: A block diagram.

1. \(L(\text{output signal})\) \quad 2. \(L(\text{input signal})\)

a. (10 points) State two definitions for “transfer function”.

b. (10 points) From Fig. 2, find the transfer function \(T_{R \rightarrow E}(s)\) from \(R(s)\) to \(E(s)\).

c. (10 points) From Fig. 2, find the transfer function \(T_{D \rightarrow E}(s)\) from \(D(s)\) to \(E(s)\).

d. (20 points) Consider the unit step input

\[ r(t) = L^{-1}(R(s)) = \begin{cases} 1 & t \geq 0; \\ 0 & 0 < t, \end{cases} \]

the unit impulse disturbance \(d(t) = L^{-1}(D(s)) = \delta(t)\), \(G_1(s) = (s + 3)\), and \(G_2(s) = \frac{1}{s+1}\) in Fig. 2. Determine the time response of \(e(t)\) under the unit step input, the unit impulse disturbance, and zero initial conditions. Hint: The Laplace transform of the unit step function, the unit impulse function, and \(e^{-at}\) are given by

\[ R(s) = L(r(t)) = \frac{1}{s}, \quad D(s) = L(\delta(t)) = 1, \quad L(e^{-at}) = \frac{1}{s + a}. \]
\[ C(s) = [D(s) + U(s)] G_2(s) \quad (1) \]

\[ U(s) = G_1(s) E(s) \quad (2) \]

\[ E(s) = R(s) - C(s) \quad (3) \]

Writing the equation in terms of \( E(s) \), \( R(s) \) and \( D(s) \),

\[ E(s) = R(s) - [D(s) + U(s)] G_2(s) \]

\[ \therefore E(s) = R(s) - [D(s) + G_1(s) E(s)] G_2(s) \]

\[ \therefore E(s) + G_1(s) G_2(s) E(s) = R(s) - G_2(s) D(s) \]

\[ \therefore E(s) = \frac{R(s)}{1 + G_1(s) G_2(s)} - \frac{G_2(s)}{1 + G_1(s) G_2(s)} D(s) \]

\[ \frac{TF_{R \rightarrow E}}{TF_{D \rightarrow E}} \frac{E(s)}{R(s)} = \frac{1}{1 + G_1(s) G_2(s)} \quad (b) \]

\[ \frac{E(s)}{D(s)} = \frac{\left( -1 \right) \times G_2(s)}{1 + G_1(s) G_2(s)} \quad (c) \]

Or just use \( \frac{F_0}{1 - L_0} \).
(a) \[ R(s) = \frac{1}{s} \],
\[ D(s) = 1, \]
\[ G_1(s) = (s+3) \]
\[ G_2(s) = \frac{1}{s+1} \]

\[ \therefore E(s) = \frac{\frac{1}{s}}{1 + \frac{(s+3)}{(s+1)}} - \frac{\frac{1}{s+1}}{1 + \frac{(s+3)}{s+1}} \]

62. Use
\[ E(s) = \frac{A}{s} + \frac{B}{s+2} \]
\[ A = \lim_{s \to 0} E(s) s^2 = \frac{1}{3} \]
\[ B = \lim_{s \to -2} E(s) s^2 = \frac{1}{3} \]

\[ = \frac{\frac{1}{3} - \frac{1}{s+1}}{1 + \frac{(s+3)}{(s+1)}} = \frac{1}{3(s+1)} \]
\[ = \frac{1}{2} \left[ \frac{1}{s(s+2)} \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{s} - \frac{1}{s+2} \right) \right] \]

\[ \therefore \]
\[ E(s) = \frac{1}{4} \left[ \frac{1}{s} - \frac{1}{s+2} \right] \]

\[ \Rightarrow E(t) = \frac{1}{4} \left[ \frac{1}{s} - \frac{1}{s+2} \right] u(t) \]