Introduction

Structural flexibility is important to the dynamics of high performance mechanical systems such as robots, boring machines, and antennas. The structural members are often modeled as a Bernoulli-Euler beam with very low natural damping. Disturbances acting on these systems result in large and long lasting vibration which can significantly degrade the system performance. Various vibration suppression techniques have been suggested to reduce this undesirable vibration [1, 2].

The vibration of Bernoulli-Euler beams is governed by a partial differential equation. Using the separation of variables method, the vibration of the structure can be represented by an infinite sum of vibration modes [3], resulting in a set of infinite-order matrices for state space control design, and a nonrational transfer function for frequency domain control design.

To design a realizable control system, the system model is often truncated, usually retaining the first n vibration modes, while the remaining terms, infinite in number, are neglected in the control design. Rationale for the truncation of higher order terms includes finite frequency bandwidth of the controller hardware and software, limited capacity of onboard computers used in real-time control applications, and use of numerical modeling techniques which generate low order models.

The actuator force on the truncated modes and the contribution of the truncated modes in the sensor output are referred to as control and observation spillover, respectively. It has been shown, theoretically [4-7] and experimentally [8, 9], that excessive spillover degrades system performance and in extreme cases destabilizes the closed-loop controlled system.

Passive control [10] and collocated direct velocity feedback [11, 12] methods have been shown to increase system damping without having the spillover problem. The ability of passive control to modify system response is limited compared with that of active control. The collocated direct velocity feedback theory cannot guarantee stability of the closed-loop system in the presence of significant dynamics in the actuators and sensors. A "low authority" controller has been suggested to limit system behavior modification and reduce spillover effects [13]. Various state space control theories have been developed for infinite dimensional systems [14-17] which use bounds on the spillover operators. These theories require the knowledge of Functional Analysis.

The work presented in the paper is based on an extended Nyquist stability criterion for distributed parameter (nonrational) systems [18]. A transfer function is formulated for the lateral vibration of a Bernoulli-Euler beam, in the form of an infinite partial fraction expansion [19-23]. Bounds, which define a tube of uncertainty, are derived for the error between truncated and actual frequency response functions. The actual system’s closed-loop stability is then evaluated using the extended Nyquist criterion and the tube of uncertainty. Numerical examples are presented to demonstrate bounds computation, and to evaluate closed-loop stability of a feedback system consisting of a proportional controller and a truncated beam model with one mode and with two modes.

Model Formulation

Consider a control system, with a single sensor and actuator pair, for feedback control of a pinned-pinned beam (Fig. 1).

![Figure 1 - The pinned-pinned Bernoulli-Euler beam showing sensor and actuator locations](image-url)
The Bernoulli-Euler equation of motion for the lateral vibration of a uniform beam including a linear damping model is
\[
EI \frac{\partial^2 z(x,t)}{\partial x^2} - \rho \frac{\partial^2 z(x,t)}{\partial t^2} + m \frac{\partial^2 z(x,t)}{\partial x^2} = a(x) u(t),
\]
(1)
where \(z(x,t)\) is the lateral displacement of an arbitrary point on the beam at any given time \(t\), \(EI\) is the bending stiffness, \(\rho\) is the damping factor, \(m\) is the mass per unit length, \(u(t)\) is the external force amplitude applied to the beam, and \(a(x)\) is the force spatial distribution. The boundary conditions corresponding to pinned ends are
\[
z(0,t) = \dot{z}(L,t) = 0,
\]
(2)
where \(L\) is the length of the beam. Without loss of generality the beam parameters \(EI, m,\) and \(L\) can be set to unity.

The solution of equations (1) and (2) can be obtained using the separation of variables method and is given by
\[
z(x,t) = \sum_{k=1}^{\infty} \phi_k(x) q_k(t),
\]
(3)
where \(\phi_k(x)\) are the modal shapes and \(q_k(t)\) are the modal amplitudes given by
\[
q_k^2(t) + 2 \gamma_k \omega_k q_k(t) + \omega_k^2 q_k(t) = u_k(t), \quad k = 1, 2, \ldots,
\]
(4)
where \(\omega_k = (k\pi)^2\) are the mode natural frequencies, \(\phi_k\) are the orthonormal mode shapes, \(\xi_k\) are the modal damping factors for undamped modes with overshoot: \(0 < \xi_1 \leq \xi_2 < 0.707,\) and \(a_k(t)\) are the modal forces. The modal forces are given by
\[
u_k(t) = \int_0^1 b(x) z(x,t) dx \Delta \alpha_k u(t), \quad k = 1, 2, \ldots
\]
(4)
The sensor output for position measurement is given by
\[
y(t) = \int_0^1 b(x) z(x,t) dx \Delta \sum_{k=1}^{\infty} \beta_k q_k(t), \quad k = 1, 2, \ldots
\]
(5)
where \(b(x)\) is the sensor spatial distribution.

The Laplace-transformed, nonrational, transfer function from the control force amplitude \(U(s)\) to the sensor output \(Y(s)\) can be derived directly from equations (3)-(5) and is given by
\[
G(s) \Delta \frac{Y(s)}{U(s)} = \sum_{k=1}^{\infty} \frac{\alpha_k \beta_k}{s^2 + 2 \gamma_k \omega_k s + \omega_k^2}.
\]
(6)
A truncated model is obtained by retaining the first \(n\) modes in equation (6) while the remaining modes, infinite in number, are neglected. This truncation results in a rational transfer function
\[
G_n(s) \Delta \sum_{k=1}^{n} \frac{\alpha_k \beta_k}{s^2 + 2 \gamma_k \omega_k s + \omega_k^2}.
\]
(7)

**Transfer Function Truncation Error Bounds**

To carry out stability analysis in the frequency domain, the transfer function (6) is converted to a nonrational frequency response function by letting \(s = j\omega\), where \(\omega\) denotes the frequency,
\[
G(j\omega) \Delta \sum_{k=1}^{\infty} \frac{\alpha_k \beta_k}{(\omega^2 - \omega_k^2) + j2\gamma_k \omega_k \omega}.
\]
(8)
The frequency response function (8) represents the steady-state frequency response of the system for a sinusoidal input. Model truncation generates transfer function errors which must be considered in the stability analysis. Define the truncation error \(E(j\omega)\) as
\[
E(j\omega) \Delta G(j\omega) - G_n(j\omega) = \sum_{k=1}^{\infty} \frac{\alpha_k \beta_k}{(\omega^2 - \omega_k^2) + j2\gamma_k \omega_k \omega}.
\]
(9)
Closed-loop stability will now be evaluated using the truncated model and bounds on the truncation error modulus. A uniform bound and a frequency dependent bound are derived below for the beam described by equations (1)-(5).

**The Uniform Bound.** The modulus of the frequency response function of a single mode can be written as
\[
|S_k(j\omega)| = \frac{\gamma_k}{\sqrt{(\omega^2 - \omega_k^2)^2 + (2\gamma_k \omega_k \omega)^2}} = \frac{\gamma_k}{\omega_k \sqrt{f_k(\omega)}}.
\]
(10)
where \(S_k(j\omega)\) denotes the \(k\)th mode frequency response function, \(\gamma_k \Delta \alpha_k \beta_k\) is a bounded constant, and
\[
f_k(\omega) \Delta [1 - \omega^2/\omega_k^2]^2 + [2\gamma_k \omega_k \omega]^2
\]
(11)
The modulus \(|S_k(j\omega)|\) has a maximum over \(\omega \in [0, \infty]\) exactly when \(f_k(\omega)\) has a minimum there. A simple algebraic rearrangement yields
\[
f_k(\omega) = \left[\frac{\omega^2 - \omega_k^2}{\omega_k^2} - 2\omega_k \omega \right]^2 + 4\gamma_k^2 (1 - \omega_k^2).
\]
(12)
and by inspection the minimum of \(f_k(\omega)\) occurs when \(\omega = \omega_k \sqrt{1 - \gamma_k^2}\). Thus the uniform bound for the modulus of the frequency response function \(S_k(j\omega)\) is
\[
|S_k(j\omega)| \leq \frac{\gamma_k}{2\omega_k \sqrt{1 - \gamma_k^2}}, \quad \omega \in [0, \infty].
\]
(13)
The uniform bound for the truncation error modulus, for a beam with natural frequencies \(\omega_k = (k\pi)^2\), is thus defined as
\[
|E(j\omega)| \leq \gamma \sum_{k=1}^{\infty} \frac{1}{\omega_k^2} \Delta R_k < \infty, \quad \omega \in (-\infty, \infty).
\]
(14)
where \(\Gamma_k \Delta \min\{2\gamma_k \sqrt{1 - \gamma_k^2}\} > 0\), and \(\gamma \Delta \max\{|\alpha_k \beta_k|\}, \quad k = 1, 2, \ldots\)

It is a common practice in control analysis to assume that the modal damping factor \(\xi_k\) is a constant for all modes. However, the uniform bound (14) allows for a wide variation in the damping ratio for different modes which agrees with experimental results [8]. To compute this bound it is sufficient to know the constants \(\epsilon_i\) and \(\epsilon_j\), giving \(\Gamma_i \Delta \min\{2\gamma_k \sqrt{1 - \gamma_k^2}\}, \quad i = 1, 2\). Therefore, this bound is robust to modal damping variations in different modes and allows flexibility in compensator design. The uniform bound provides a constant nonzero bound for all frequencies.

**The Frequency Dependent Bound.** A bound that approaches zero as the frequency approaches infinity can be derived. Consider the modulus of the \(k\)th mode frequency response function rearranged to
\[
|T_k(j\omega)| = \frac{\gamma_k}{\omega \omega_k \sqrt{h_k(\omega)}},
\]
(15)
where
\[
h_k(\omega) = \left[\frac{\omega^2 - \omega_k^2}{\omega_k^2}\right]^2 + 4\gamma_k^2.
\]
(16)
The modulus \(|T_k(j\omega)|\) can be bounded above using the inequality \(17\) \(\gamma_k \leq \min\{1/\omega_k \omega_k \min\{h_k(\omega)\}, \omega \in [0, \infty].\)

The minimum of \(h_k(\omega)\) occurs when \(\omega = \omega_k\) giving
\[
|T_k(j\omega)| \leq \frac{\gamma_k}{2\omega_k \omega_k}, \quad \omega \in [0, \infty].
\]
(17)
The frequency dependent bound for the truncation error
modulus, for a beam with natural frequencies \( \omega_n = (k \pi)^2 \), is thus defined as

\[
|E(\omega)| \leq \frac{1}{1 - \omega^2} \sum_{k=1}^{\infty} \frac{1}{\omega_n^2} \Delta R_n(\omega) < \infty, \quad \omega \in (-\infty, \infty),
\]

where \( \Gamma_2 = \min \{ 2 \gamma \} = 2 \epsilon_1 > 0 \), and \( \Delta \max \{ \delta_1, \delta_2 \}, k = 1, 2, \ldots \). This bound increases for decreasing frequencies below one and approaches zero as \( \omega \to \infty \).

The smaller of the bounds \( R_1 \) and \( R_2(\omega) \) can be used over different frequency ranges to produce a smaller overall bound (Fig. 2). Note that as more modes (terms) are included in the truncated model, both bounds decrease with limit zero as \( n \to \infty \).

**Graphical Interpretation of the Error Bounds.** The nonrational frequency response function \( G(\omega) \) is within the error bounds \( R_1 \) and \( R_2(\omega) \) of the truncated (rational) frequency response function \( G_\alpha(\omega) \). At each frequency, \( G(\omega) \) is within circles of radii \( R_1 \) and \( R_2(\omega) \), centered at the point \( G_\alpha(\omega) \) (Fig. 3). Therefore, \( G(\omega) \) always lies within the smaller of the two circles. That circle represents the uncertainty due to the order truncation.

The polar plot for \( G(\omega) \) over a range of frequencies has a similar interpretation. The polar plot \( G_\alpha(\omega) \) is drawn together with error circles associated with each frequency in that range. The union of all the smaller error circles defines two plots: an interior boundary and an exterior boundary. The polar plot of \( G(\omega) \) is then found within that tube of uncertainty, enclosed by the interior and exterior boundaries (Fig. 3). The tube of uncertainty represents a bound on model truncation error in the frequency domain. The tube of uncertainty corresponds, in an abstract sense, to spillover bounds [5] and Gershgorin discs [25] in the time domain.

**Example 1: Error Bounds Calculation.** Consider a control system (Fig. 4) where \( G(s) \) corresponds to a nonrational transfer function described by equations (1)-(5). For this example we set the bar parameters \( E, \alpha \), and \( L_i \) to unity, and allow modal damping factors between \( \epsilon_1 = 0.005 \) and \( \epsilon_2 = 0.5 \). The uniform bound, \( R_1 \), and the frequency dependent bound, \( R_2(\omega) \), are calculated using equations (14) and (18), respectively. For the above system we have: \( \gamma = 2 \), \( \Gamma_1 = \Gamma_2 = 0.01 \). For a truncated model which consists of the first mode (\( n = 1 \) alone):

\[
R_1 = 200 \sum_{k=2}^{\infty} \frac{1}{(k\pi)^2} = 0.1692
\]

and

\[
R_2(\omega) = \frac{200}{\omega} \sum_{k=2}^{\infty} \frac{1}{(k\pi)^2} = \frac{13.18}{\omega}. \tag{19b}
\]

For a truncated model which consists of the first ten modes (\( n = 10 \)):

\[
R_1 = 200 \sum_{k=11}^{\infty} \frac{1}{(k\pi)^2} \approx 0.0000373 \tag{20a}
\]

and

\[
R_2(\omega) = \frac{200}{\omega} \sum_{k=11}^{\infty} \frac{1}{(k\pi)^2} = \frac{0.0102}{\omega} \tag{20b}
\]

**Stability Analysis.**

An extended Nyquist stability criterion will be applied to feedback control of the pinned-pinned beam. The following results can be applied to many closed-loop systems consisting of linear actuator and sensor transfer functions modeled in the forward and/or feedback loop.

Consider a closed-loop control system consisting of a forward-loop controller \( G_\alpha(\omega) \), the beam's transfer function \( G(s) \), and a negative unity feedback loop \( G_f = 1 \) (Fig. 4). To begin Nyquist stability analysis [26], consider the open-loop frequency response function:

\[
\]

The open-loop transfer function \( Q(s) \) is assumed to be strictly proper in the closed right-half s-plane. The Nyquist plot for \( Q(\omega) \) is found within the tube of uncertainty about the truncated frequency response function \( Q_\alpha(\omega) = G_\alpha(\omega)G(\omega) \). The truncation error modulus bound for the open-loop frequency response function is

\[
|Q(\omega) - Q_\alpha(\omega)| = |G_\alpha(\omega)E(\omega)| \leq |G_\alpha(\omega)|R_1,
\]

where \( i = 1, 2 \),

\[
|Q(\omega)| \leq |Q_\alpha(\omega)| + |Q(\omega) - Q_\alpha(\omega)| \leq |Q_\alpha(\omega)| + |Q(\omega)|R_1,
\]

provided that \( G_\alpha(\omega) \) is bounded. The tube of uncertainty is
defined by the smaller error circle of radius \(|G_1(j\omega)R_0|\) obtained for each point in the range \(\omega\).

Because the Nyquist plot can only be computed up to some finite frequency, say \(\omega^*\), we must define an additional bound circle which contains the Nyquist plot of \(Q_1(j\omega)\) for all frequencies higher than \(\omega^*\). The radius of this additional bound obtained for equation (21) is

\[
R_4(\omega^*) \Delta \max \left\{|G_i(j\omega)| + R_0 \right\}, \quad i = 1, 2.
\]

(23)

When \(R_4(\omega^*)\) is large enough that no stability conclusion can be drawn, then either the order of \(G_k(j\omega)\) or the maximum computed frequency \(\omega^*\) must be increased until a conclusion can be drawn. An example for the above condition is whenever \(R_4(\omega^*)\) is greater than unity. Note that the infinite semi-circle portion of the Nyquist plot maps into the \((0, 0)\) point since \(Q(s)\) is strictly proper.

The classical Nyquist stability criterion [26] can be used only if the open-loop transfer function is rational, which is not the case in our system. An extension of this criterion, which covers non-rational transfer functions such as the one in equation (6) has been developed in [18]. The closed-loop stability of the system shown in Fig. 4 with a stable, non-rational, open-loop transfer function \(Q(s)\) describing the system in (1)-(5) can now be determined using the tube of uncertainty and the extended Nyquist stability criterion [18]. The following three possibilities can occur:

(i) whenever the point \(-1,0\) is not encircled by the tube of uncertainty, then the closed-loop system is asymptotically stable,

(ii) whenever the point \(-1,0\) is encircled by the tube of uncertainty, then the closed-loop system is unstable,

(iii) whenever the point \(-1,0\) lies inside the tube of uncertainty, then stability is an open question.

For an encirclement of \(-1,0\) by the tube, it is necessary that the truncated Nyquist plot encircles \(-1,0\) point. Note that encirclement in case (ii) will necessarily be in the clockwise direction. In practice, the compensator is chosen such that \(Q_s(s)\) yields a stable closed-loop system. Therefore, the particular cause by truncation is that the \(-1,0\) point lies inside the tube of uncertainty as in case (iii). Retaining more modes in the truncated model may result in case (i).

**Example 2: Stability Verification.** Consider the feedback system shown in Fig. 4, a point actuator located at \(x_h = 1/7\) so that \(a_1 = \sqrt{2}\sin(\pi/7)\), and a point sensor located at \(x_s = 5/7\) so that \(b_1 = \sqrt{2}\sin(\pi/7)\). The truncated model consisting of the first mode only and \(G_1 = 0.005\) is given by

\[
G_1(s) = \frac{0.6784}{s^2 + 0.0987s + 97.4091}.
\]

(24)

The error bounds are identical to those in equations (19a) and (20a). For the sake of demonstrating the stability criterion, a proportional controller

\[
G_2(s) = K_P,
\]

and a negative unity feedback loop \(G_1 = 1\) were chosen. The Nyquist plot of the open-loop system \(G_1G_2(j\omega)\) (up to 12 rad/s) and the tube of uncertainty are shown in Fig. 5, for a gain \(K_P = 1.0\). The circle bound for \(\omega > 12\) rad/s is \(R_1 = 0.184\) in this example. Because the exterior boundary of the tube of uncertainty does not encircle the point \(-1,0\) and the system is open-loop stable, we conclude that the closed-loop system consisting of the non-rational transfer function (6) and the controller (25) is asymptotically stable.

![Fig. 5 Nyquist plot of the system in example 2 with \(n = 1\)](image)

The condition for stability can be found, using our criterion, for different gains \(K_P\):

\[
0 < K_P < 2.55 \quad \text{asymptotically stable (case i),} \quad 2.55 \leq K_P \quad \text{stability is an open question (case iii).}
\]

In this example, the open-loop truncated frequency response function will not encircle the \((-1,0)\) point for any choice of \(K_P > 0\), hence, the tube of uncertainty will never encircle the \((-1,0)\) point.

The consequences of retaining more modes in the truncated model are important. Let us consider the above feedback system with \(G_2 = 1.0\) and \(G_1 = 1\) and a truncated model consisting of the first two vibration modes

\[
G_1(s) = \frac{0.6784}{s^2 + 0.0987s + 97.4091} + \frac{-1.5245}{s^2 + 0.3947s + 1588.54}.
\]

(26)

As expected, the new error bounds are smaller: \(R_1 = 0.0408\) and \(R_1(\omega) = 8.106/\omega\). The new Nyquist plot in Fig. 6 is plotted up to \(\omega = 42\) rad/s and the error bound for \(\omega > 42\) rad/s is \(R_1 = 0.05\). Increasing the number of modes in \(G_1(s)\) always decreases the radius of the tube of uncertainty at each frequency.

Adding a term to \(G_1(s)\) results in a drastic change in the Nyquist plot of the open-loop frequency response function as indicated by equation (26) and Fig. 6. Because the new tran-
cated Nyquist plot crosses the negative real axis there is a possibility that the tube of uncertainty will encircle the \((-1, 0)\) point for some large gain \(K_p\). It can be shown that the closed-loop system with \(G_2\) is asymptotically stable when \(0 < K_p < 5.62\). This maximum stable gain range is larger compared with the previous gain range \(0 < K_p < 2.55\) obtained for \(G_1\). Increasing the number of modes in the truncated model always changes the truncated frequency polar plot as well as the size of the tube of uncertainty.

**Summary**

Model truncation errors and the resulting effects (spillover) on closed-loop stability can be predicted for feedback control of the Bernoulli-Euler beam using the frequency domain stability criterion presented in this paper. In this criterion, the stability was verified for the nonrational closed-loop system, although the controller was designed for a truncated model. The stability was derived using simple graphical tools, the tube of uncertainty and the Nyquist criterion. The tube of uncertainty was obtained from truncation bound circles, easily computed for a pinned-pinned beam. The criterion provides a practical stability analysis method for feedback control of the Bernoulli-Euler beam and has important implications for control of other distributed parameter systems.

**References**