1. (1) Try \( V(x) = \frac{1}{2}(x_1^2 + x_2^2) \).

\[
\dot{V}(x) = x_1(-x_1 + x_2^2) - x_2^2 = -x_1^2 - x_2^2 + x_1x_2
\]

In the neighborhood of the origin, the term \(-x_1^2 - x_2^2\) dominates. Hence, the origin is asymptotically stable. Moreover

\[
x_2(t) = e^{-t}x_{20} \Rightarrow x_1(t) = e^{-t}x_{10} + \int_0^t e^{-(t-s)}e^{-2s} \, ds \, x_{20} = e^{-t}x_{10} + [e^{-t} - e^{-2t}]x_{20}
\]

For all \( x_0, \; x(t) \to 0 \) as \( t \to \infty \), which implies that the origin is globally asymptotically stable.

(2) Try \( V(x) = ax_1^2 + bx_2^2, \; a > 0, \; b > 0 \).

\[
\dot{V}(x) = 2ax_1(x_1-x_2)(x_1^2+x_2^2-1) + 2bx_2(x_1+x_2)(x_1^2+x_2^2-1) = 2[a(x_1-x_2)+bx_2(x_1+x_2)](x_1^2+x_2^2-1)
\]

Let \( a = b \).

\[
\dot{V}(x) = -2a(x_1^2 + x_2^2)[1 - (x_1^2 + x_2^2)]
\]

For \( x_1^2 + x_2^2 < 1 \), \( \dot{V}(x) \) is negative definite. Hence, the origin is asymptotically stable. It is not globally asymptotically stable since there other equilibrium points on the unit circle.

(3) Try \( V(x) = \frac{1}{2}(ax_1^2 + bx_2^2), \; a > 0, \; b > 0 \).

\[
\dot{V}(x) = ax_1(-x_1 + x_2^2) + bx_2(-x_2 + x_1) = -ax_1^2 + bx_1x_2 - bx_2^2 + ax_2^2 = -x^TQx + ax_1^3x_2
\]

where \( Q = \begin{bmatrix} a & -0.5b \\ -0.5b & b \end{bmatrix} \). The matrix \( Q \) is positive definite when \( ab - b^2/4 > 0 \). Choose \( b = a = 1 \). Near the origin, the quadratic term \(-x^TQx\) dominates the fourth-order term \( x_1^3x_2 \). Thus, \( \dot{V}(x) \) is negative definite and the origin is asymptotically stable. It is not globally asymptotically stable since there are other equilibrium points at \((1,1)\) and \((-1,-1)\).

2. Consider \( V(x) = x_1^2 + x_2^2 \) as a Lyapunov function candidate.

\[
\dot{V}(x) = 2x_1^2(k^2 - x_1^2 - x_2^2) + 2x_2^2(x_1^2 + x_2^2 + k^2) - 2x_1x_2(x_1^2 + x_2^2 + k^2) + 2x_2^2(k^2 - x_1^2 - x_2^2)
\]

\[
= 2(x_1^2 + x_2^2)(k^2 - x_1^2 - x_2^2)
\]

(a) \( k = 0 \Rightarrow \dot{V}(x) = -2(x_1^2 + x_2^2)^2 \)

The origin is globally asymptotically stable.

(b) \( k \neq 0 \Rightarrow \dot{V}(x) > 0, \; \text{for} \; 0 < x_1^2 + x_2^2 < k^2 \)

By Chetaev’s theorem, the origin is unstable.

3. Try \( g(x) = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix} \). To meet the symmetry requirement, take \( \gamma = \beta \).

\[
\dot{V}(x) = (\alpha x_1 + \beta x_2)x_2 - (\beta x_1 + \delta x_2)(x_1 + x_2)\sin(x_1 + x_2)
\]

Take \( \delta = \beta \).

\[
\dot{V}(x) = -\beta x_1^2 + (\alpha - 2\beta)x_1x_2 - \beta(x_1 + x_2)\sin(x_1 + x_2)
\]

Taking \( \alpha = 2\beta \) and \( \beta > 0 \) yields

\[
\dot{V}(x) = -\beta x_1^2 - \beta(x_1 + x_2)\sin(x_1 + x_2)
\]
which is negative definite in the set \(\{|x_1 + x_2| < \pi\}\). Now
\[
g(x) = \beta \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \overset{\text{def}}{=} Px \implies V(x) = \int_0^x g^T(y) \, dy = \frac{1}{2} x^T P x
\]
where \(P\) is positive definite.

4. For \(|2x_1 + x_2| \leq 1\), we have
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x
\]
This matrix is Hurwitz. Hence, the origin is asymptotically stable. It is not globally asymptotically stable because there is another equilibrium point at \((1,0)\).

5. Take \(V(x) = -\frac{1}{8} x_1^6 + \frac{1}{4} x_2^4\).
\[
\dot{V}(x) = -x_1^5 \dot{x}_1 + x_2^3 \dot{x}_2 = x_1^6 + x_2^6 - x_1^5 x_2^6 + x_2^3 x_1^6
\]
Near the origin
\[
|-x_1^5 x_2^6 + x_2^3 x_1^6| \leq \frac{1}{2} (x_1^6 + x_2^6)
\]
Hence
\[
\dot{V}(x) \geq \left(1 - \frac{1}{2}\right) (x_1^6 + x_2^6)
\]
which shows that \(\dot{V}(x)\) is positive definite. Application of Chetaev’s theorem shows that the origin is unstable.

6. Since \(zg(z) > 0\) for \(z \neq 0\), we have \(g(z) > 0\) for \(z > 0\) and \(g(z) < 0\) for \(z < 0\). Since \(g(z)\) is continuous, \(g(0) = 0\). At the origin \((z = 0, y = 0)\), \(\sum a_i y_i = 0, h(\cdot, \cdot) y_i = 0, \) and \(g(0) = 0\). Hence, the origin is an equilibrium point. Consider the Lyapunov function candidate
\[
V(z, y) = \alpha \int_0^z g(s) \, ds + \sum_{i=1}^m \beta_i y_i^2, \quad \alpha > 0, \quad \beta_i > 0
\]
\(V(z, y)\) is positive definite and radially unbounded.
\[
\dot{V} = \alpha g(z) \dot{z} + 2 \sum_{i=1}^m \beta_i y_i \dot{y}_i = \sum_{i=1}^m (-\alpha a_i + 2 \beta_i b_i) y_i g(z) - 2 \sum_{i=1}^m \beta_i h(z, y) y_i^2
\]
Choose \(\beta_i = \alpha a_i / 2 b_i\), to obtain
\[
\dot{V} = -2 \sum_{i=1}^m \beta_i h(z, y) y_i^2 \leq 0, \quad \forall (z, y)
\]
Thus \(\dot{V}\) is negative semidefinite.
\[
\dot{V} = 0 \Rightarrow y_i(t) \equiv 0, \quad \forall i \Rightarrow \dot{y}_i(t) \equiv 0, \quad \forall i \Rightarrow g(z(t)) \equiv 0 \Rightarrow z(t) \equiv 0
\]
By LaSalle’s theorem (Corollary 4.2), the origin is globally asymptotically stable.

7.
\[
0 = x_2 \\
0 = -x_1 - x_2 \text{sat}(x_2^2 - x_3^2) \Rightarrow x_1 = 0 \\
0 = x_3 \text{sat}(x_2^2 - x_3^2) \Rightarrow x_3 \text{sat}(-x_3^2) = 0 \Rightarrow x_3 = 0
\]
Hence, the origin is the unique equilibrium point. Consider
\[
V(x) = x^T x = x_1^2 + x_2^2 + x_3^2 \\
\dot{V}(x) = 2[x_1x_2 - x_1x_2 - x_2^2 \text{sat}(x_2^2 - x_2) + x_3^2 \text{sat}(x_2^2 - x_2^3)] = -2(x_2^2 - x_2^3) \text{sat}(x_2^2 - x_2^3) \leq 0
\]
\[\dot{V} \text{ is negative semidefinite.} \]
\[\dot{V} = 0 \Rightarrow x_2^2(t) = x_3^2(t) \Rightarrow \dot{x}_3(t) \equiv 0 \]
Hence, both \(x_2(t)\) and \(x_3(t)\) must be constants. This implies that \(\dot{x}_2(t) \equiv 0\). From the second state equation we conclude that \(x_1(t) \equiv 0\). Then, the first state equation implies that \(x_2(t) \equiv 0\). Consequently, \(x_3(t) \equiv 0\). By LaSalle’s theorem (Corollary 4.2), the origin is globally asymptotically stable.

8. \(V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\) is positive definite and radially unbounded.
\[
\dot{V}(x) = x_1^3[-kh(x)x_1 + x_2] + x_2[-h(x)x_2 - x_1^3] = -kx_1^3h(x) - x_2^2h(x)
\]
**(1)** \(k > 0, h(x) > 0 \forall \ x \in D\). In this case \(\dot{V}(x)\) is negative definite. Hence, the origin is asymptotically stable. 

**(2)** \(k > 0, h(x) > 0 \forall \ x \in R^2\). In this case \(\dot{V}(x) < 0, \forall \ x \neq 0\). Hence, the origin is globally asymptotically stable. 

**(3)** \(k > 0, h(x) < 0 \forall \ x \in D\). In this case \(\dot{V}(x)\) is positive definite. Hence, by Chetaev’s theorem, the origin is unstable. 

**(4)** \(k > 0, h(x) = 0 \forall \ x \in D\). In this case \(\dot{V}(x) = 0\). Hence, the origin is stable. It is not asymptotically stable because trajectories starting on the level surface \(V(x) = c\) remain on the surface for all future time. 

**(5)** \(k = 0, h(x) > 0 \forall \ x \in D\). In this case \(\dot{V}(x) = -x_2^2h(x) \leq 0, \forall \ x \in D\). Moreover 
\[
\dot{V}(x) = 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0
\]
Hence, by LaSalle’s theorem (Corollary 4.1), the origin is asymptotically stable. 

**(6)** \(k = 0, h(x) > 0 \forall \ x \in R^2\). The same as part (5), except that all the conditions hold globally. Hence, the origin is globally asymptotically stable.

9. **(a)**
\[
0 = -x_1 + g(x_3), \quad 0 = -g(x_3), \quad 0 = -ax_1 + bx_2 - cx_3
\]
From the properties of \(g(\cdot)\) we know that \(g(x_3) = 0\) has an isolated solution \(x_3 = 0\). Substituting \(x_3 = 0\) in the foregoing equations we obtain \(x_1 = x_2 = 0\). Hence, the origin is an isolated equilibrium point.

**(b)**
\[
V(x) = \frac{a}{2}x_1^2 + \frac{b}{2}x_2^2 + \int_0^{x_3} g(y) \ dy \\
\dot{V}(x) = ax_1[-x_1 + g(x_3)] - bx_2(g(x_3)) + g(x_3)[-ax_1 + bx_2 - cg(x_3)] = -ax_1^2 - cx_3^2 \leq 0
\]
\[\dot{V}(x) = 0 \Rightarrow x_1(t) \equiv 0 \text{ and } x_3(t) \equiv 0 \Rightarrow \dot{x}_3(t) \equiv 0 \]
From the third state equation we see that \(x_2(t) \equiv 0\). Hence, by LaSalle’s theorem (Corollary 4.1), the origin is asymptotically stable. 

**(c)** To conclude that the origin is globally asymptotically stable, we need to know that \(V(x)\) is radially unbounded. But this is not guaranteed since 
\[
yg(y) > 0, \forall |y| \neq 0 \Rightarrow \int_0^x g(y) \ dy \to \infty \text{ as } |x| \to \infty.
\]
Consider, for example, \( g(y) = (1 - e^{-|y|})e^{-|y|}\text{sgn}(y) \). For \( x > 0 \), we have

\[
\int_0^x (1 - e^{-y}) e^{-y} \, dy = 1 - e^{-x} - \frac{1}{2}(1 - e^{-2x}) \to \frac{1}{2} \quad \text{as} \quad x \to \infty
\]

Thus we cannot conclude that the origin is globally asymptotically stable.

10.

\[
V(x) = 2a(1 - \cos x_1) + kx_1^2 + x_2^2 + px_3^2 \geq kx_1^2 + x_2^2 + px_3^2
\]

Hence, \( V \) is positive definite and radially unbounded.

\[
\dot{V}(x) = 2(-dx_2^2 - cx_2x_3 - px_3^2 + px_2x_3)
\]

Taking \( p = c \), we obtain

\[
\dot{V} = -2dx_2^2 - 2px_3^2 \leq 0, \quad \forall \ x \in R^3
\]

\[
\dot{V} \equiv 0 \Rightarrow x_2(t) \equiv 0 \& x_3(t) \equiv 0 \Rightarrow a \sin x_1(t) + kx_1(t) \equiv 0
\]

Since \( k > a \),

\[
a \sin x_1(t) + kx_1(t) \equiv 0 \Rightarrow x_1(t) \equiv 0
\]

Using LaSalle's theorem (Corollary 4.2), we conclude that the origin is globally asymptotically stable.

11.

\[
\begin{align*}
0 &= \frac{1}{1 + x_3} - x_1 \\
0 &= x_1 - 2x_2 \Rightarrow x_1 = 2x_2 \\
0 &= x_2 - 3x_3 \Rightarrow x_3 = \frac{1}{3}x_2
\end{align*}
\]

Substitution of \( x_1 \) and \( x_3 \) in the first equation yields

\[
2x_2^2 + 6x_2 - 3 = 0 \Rightarrow x_2 = \frac{-3 \pm \sqrt{15}}{2}
\]

Thus there is only one equilibrium point in the region \( x_i \geq 0 \); namely,

\[
x_1 = \sqrt{15} - 3, \quad x_2 = \frac{\sqrt{15} - 3}{2}, \quad x_3 = \frac{\sqrt{15} - 3}{6}
\]

\[
\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 & \frac{-1}{(1+x_3)^2} \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix} \bigg|_{x_3 = \frac{\sqrt{15} - 3}{6}} = \begin{bmatrix} -1 & 0 & -0.7621 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}
\]

The eigenvalues are \(-1.3671 \pm j \ 0.449\) and \(-3.2657\). Hence, the equilibrium point is asymptotically stable.

12. (1)

\[
\begin{align*}
0 &= -x_1 + x_2 \Rightarrow x_1 = x_2 \\
0 &= (x_1 + x_2) \sin x_1 - 3x_2
\end{align*}
\]

Thus

\[
x_1(2 \sin x_1 - 3) = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0
\]
Hence, the origin is the unique equilibrium point.

\[
\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ \sin x_1 + (x_1 + x_2) \cos x_1 & \sin x_1 - 3 \end{bmatrix}; \quad A = \frac{\partial f}{\partial x} \bigg|_{x=0} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}
\]

\(A\) is Hurwitz; hence, the origin is asymptotically stable. To show global asymptotic stability, let \(V(x) = \frac{1}{2}(x_1^2 + x_2^2)\).

\[
\dot{V}(x) = -x_1^3 + x_1 x_2 (1 + \sin x_1) - (3 - \sin x_1) x_2^2 \leq -x_1^2 + 2|x_1| |x_2| - 2x_2^2
\]

Hence, the origin is globally asymptotically stable.

(2)

\[
0 = -x_1^3 + x_2
\]

\[
0 = -ax_1 - bx_2 \Rightarrow x_2 = -\frac{b}{a}x_1 \Rightarrow -x_1(x_1^2 + a/b) = 0 \Rightarrow x_1 = 0
\]

Hence, the origin is the unique equilibrium point.

\[
\frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\ -a & -b \end{bmatrix}; \quad A = \frac{\partial f}{\partial x} \bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}
\]

\(A\) is Hurwitz; hence, the origin is asymptotically stable. To show global asymptotic stability, let \(V(x) = \frac{1}{2}(x_1^2 + \alpha x_2^2), \alpha > 0\).

\[
\dot{V}(x) = -x_1^4 + x_1 x_2 (1 - a\alpha) - ba x_2^2
\]

Taking \(\alpha = 1/a\), we obtain

\[
\dot{V}(x) = -x_1^4 - \frac{b}{a} x_2^2 < 0, \quad \forall x \neq 0
\]

Hence, the origin is globally asymptotically stable.

13. Take \(V(x) = \frac{1}{2}(x_1^2 + x_2^2)\).

\[
\dot{V} = x_1[-x_1^3 + \alpha(t)x_2] + x_2[-\alpha(t)x_1 - x_2^3] = -x_1^4 - x_2^4
\]

Hence, the origin is globally uniformly asymptotically stable. Linearization at the origin yields

\[
A(t) = \begin{bmatrix} 0 & \alpha(t) \\ -\alpha(t) & 0 \end{bmatrix}
\]

The origin of the linear system is not exponentially stable since, with \(V(x) = \frac{1}{2}(x_1^2 + x_2^2)\), we have \(\dot{V} = 0\) which implies that \(V(x(t))\) is constant along the solution. Thus, \(x(t)\) does not converge to the origin as \(t\) tends to infinity. From Theorem 4.15, we conclude that the origin of the nonlinear system is not exponentially stable.

14. View this system as a perturbation of the autonomous system

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2
\]

The matrix \(A\) of the autonomous system is Hurwitz. Find a Lyapunov function \(V(x) = x^T P x\) for the autonomous system by solving the Lyapunov equation \(PA + A^T P = -I\), to obtain \(P = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\).
Now use $V(x) = x^T P x = \frac{3}{2} x_1^2 + x_1 x_2 + x_2^2$ as a Lyapunov function candidate for the perturbed system with $b \neq 0$.

\[
\dot{V} = -x_1^2 - x_2^2 - b \cos t \cdot x_1 x_2 - 2b \cos t \cdot x_2^2 \leq -x_1^2 - x_2^2 + |b| \cdot |x_1| \cdot |x_2| + 2|b| \cdot |x_2|^2
\]

\[
= - \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -|b|/2 \\ -|b|/2 & 1 - 2|b| \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}
\]

The right hand side is negative definite if $(1 - 2|b| - b^2/4) > 0$, which is the case if $|b| < b^*$. Taking $b^* = 0.472$, we conclude that the origin is exponentially stable for all $|b| < b^*$.

15. Try $V(x) = \frac{1}{2} (x_1^2 + x_2^2)$.

\[
\dot{V} = x_1 x_2 - g(t) x_1^2(x_1^2 + x_2^2) - x_1 x_2 - g(t) x_2^2(x_1^2 + x_2^2) = -g(t)(x_1^2 + x_2^2)^2 \leq -4kV^2(x)
\]

Hence, $\dot{V}(t, x)$ is negative definite and the origin is uniformly asymptotically stable. From the inequality $\dot{V} \leq -4kV^2$, we cannot conclude exponential stability. Let us try linearization.

\[
A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0} = \begin{bmatrix} -3g(t)x_1^2 - g(t)x_2^2 & 1 - 2g(t)x_1x_2 \\ -1 - 2g(t)x_1x_2 & -3g(t)x_2^2 - g(t)x_1^2 \end{bmatrix} \bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

$A$ is a constant matrix that is not Hurwitz. Hence, by Theorem 4.15, we conclude that the origin is not exponentially stable.

16. Let $A_1 = \frac{\partial f}{\partial x}(0)$ be the linearization of (1). To find the linearization of (2), set $g(x) = h(x)f(x)$. Then

\[
\frac{\partial g_i}{\partial x_j} = h(x) \frac{\partial f_i}{\partial x_j} + \frac{\partial h}{\partial x_j} f_i(x)
\]

Hence

\[
\frac{\partial g_i}{\partial x_j}(0) = h(0) \frac{\partial f_i}{\partial x_j}(0) + \frac{\partial h}{\partial x_j}(0) f_i(0) = h(0) \frac{\partial f_i}{\partial x_j}(0)
\]

Hence

\[
A_2 = \frac{\partial g}{\partial x}(0) = h(0) A_1
\]

Since $h(0) > 0$, $A_1$ is Hurwitz if and only if $A_2$ is Hurwitz. By Theorem 4.15

(1) is exp. stable $\iff$ $A_1$ is Hurwitz $\iff$ $A_2$ is Hurwitz $\iff$ (2) is exp. stable

17. It is not input-to-state stable because with $u = 0$ the origin is not globally asymptotically stable (notice that the unforced system has equilibrium points on the unit circle).