Abstract—A humanoid neck design with low motion noise yields a cable-driven parallel manipulator to imitate two axis rotational motion of a human neck. The fixed base and moving platform of the manipulator are connected by four cables and a column compression spring. The four cables are actuated separately, while the spring can support weight on the moving platform. Although similar manipulators exist in literature, the analysis for them is scarce. The difficulty is that while traditional parallel manipulator usually has a rigid kinematic chain as the spine, this manipulator has a flexible spring. With the spring’s lateral bending motion, new approach must be adopted to solve the kinematics. In this paper, we propose a method which combines the kinematics with the statics to solve them simultaneously. We parameterize the posture of the moving platform with four parameters and consider one of them as parasitic motion. Using the spring bending equation, we can obtain the parasitic motion and solve the inverse position problem. The analysis in this paper provides a novel way to analyze parallel manipulator with a spring spine, and it can be applied to other manipulators with flexible spines.

I. INTRODUCTION

The manipulator analyzed in this paper comes from a humanoid neck system with low motion noise requirement to investigate the acoustic issues during the motion. In [1], we have described our low motion noise humanoid neck system. The major mechanism in the system is shown in Fig. 1. The fixed plate is connected to a torso which is not shown in the figure, while the mobile plate is supporting the head. Four evenly distributed cables are used to control the orientation of the mobile plate. A compression coil spring, connecting the centers of fixed plate and mobile plate, has similar function of the cervical spine which can support the head and facilitate its motion. The mobile plate in the figure has two rotational degree of freedom (DOF) which can bend the head in a specified direction with desired amplitude.

Cable-driven mechanisms have recently attracted a lot of research interests. Compared to traditional robots, they are relatively simple in form and possess some valuable characteristics such as large workspace, low inertia, high payload to weight ratio, transportability, reconfigurability, and fully remote actuation [2]. Research has been performed based on practical implementation, and a large number of cable-driven robots have been developed such as the RoboCrane for moving heavy loads over large workspace [3], the WARP manipulator for assembling of lightweight objects [4], the large FAST system for a radio telescope receiver [5][6], and the NIMS3D cabled robot for sensing applications [7]. There are a lot of prior studies on kinematics and workspace analysis [8], force-closure analysis [9][10], optimal cable tension distribution [11], and dynamics and control [5][12].

The cable driven manipulator in our low motion noise head shown in Fig. 1 belongs to the category of Incompletely Restrained Positioning Mechanism (IRPM) according to Ming [13]. For IRPM, the manipulator can not maintain its own rigidity and needs external load to make all cables taut. Three types of external load are usually adopted: gravity force, buoyancy force, and artificial external force produced by spine-like element. Our mechanism belongs to the case using the spring as the spine element. Although this idea is not new, the analysis for such a mechanism is scarce. In the iCub head design, the authors just list the cable driven spring based mechanism structure without providing any analysis [14]. To the best of our knowledge, the only analysis for such a mechanism is presented in the James head design [15], where the inverse kinematics is obtained. The analysis, however, relies on assumptions of a constant spring length and a constant spring curvature, which cannot hold all the time. In fact, the deformation of the spring should be treated as a lateral bending problem. The first analysis considering spring bending is given by us in [1], where the statics is obtained. The derivation, however, is for a simplified case when only two of the four cables are actuated simultaneously. This paper aims to, based on lateral bending model of the compression spring, give a general solution to the kinematics and statics when all the four cables are
actuated independently. It will be shown that it is different from traditional analysis with separate kinematics and statics. Instead, we need to combine them together to obtain the results because of the lateral bending of the spring.

II. DESCRIPTION OF THE MANIPULATOR

The general configuration of the mechanism analyzed in this paper is shown in Fig. 2. There are four main components of the mechanism:

Fixed base: It is the fixed part of the mechanism to which a fixed coordinate frame $OXYZ$ is attached. The origin of the frame is at the bottom center of the spring, $X$ axis is along $OA_1$, $Y$ axis is along $OA_2$, and $Z$ axis is determined by the right hand rule.

Moving platform: It is the moving part of the mechanism to which the head will be mounted. A body frame $oxy$ is attached to this moving platform, with the origin $o$ at the top center of the spring, $x$ axis is along $oB_2$, $y$ axis is along $oB_1$, and $z$ axis is determined by the right hand rule.

Cables: Four flexible cables with negligible mass and diameter are connected to the moving platform at points $B_i (i = 1, 2, 3, 4)$ and pulled from the fixed base at point $A_i$. Both $A_i$ and $B_i$ are equidistance to each other on the circle with radii $|OA_i| = a$ and $|OB_i| = b$ with respect to the center $O$ and $o$, respectively. Denote the force value along the cable as $F_i$, and the cable length between two plates as $l_i$. $u_i$ is the unit vector for the force direction point to the fixed base. We use the bold letter to denote the vector variable, for example, $T_iu_i = T_iu_i$. Similar expressions are applied for other variables.

Spine: The compression spring produces a force/torque between the fixed base and the moving platform to support the robot head and facilitate the head motion. Note that since the spine is rigidly connected to both the fixed base and the moving platform, the tangent vectors for two ends of the spring are always perpendicular to fixed base plane and moving platform plane. The only assumption for the spring is that when the cables are actuated, the spring will bend in a plane. Because this assumption will be frequently referred to, we state it as follows:

Assumption 1 The center curve of the spring is always in a plane.

This plane, shown in Fig. 2, is formed by $O$, $o$, and $o'$, where $o'$ is the vertical projection of $o$ onto the fixed base. In this plane, a planar body frame $Ost$ is attached to this spring. The origin is the same as the fixed frame $OXYZ$, $t$ axis is the same as the $Z$ axis in $OXYZ$, and $s$ axis is along $Oo'$.

As defined by the IFToMM, the DOF is the number of independent coordinates needed to define the configuration of a mechanism [16]. According to this definition, the mechanism has four DOF because we need four parameters to define the configuration of the moving platform: $\theta_s$, angle between axis $s$ and axis $X$ (rotation direction); $\theta_p$, angle between the fixed base plane and the moving platform plane (rotation amplitude); $t_0$, $t$ coordinate for point $o$ in the spring attached body frame $Ost$ (vertical length of the bending spring); $s_0$, $s$ coordinate for point $o$ in body frame $Ost$ (lateral translation for the bending spring).

However, as we will show later, there exists only three independent parameters under assumption 1. Since $\theta_s$ and $\theta_p$ are the variables we want to control, we need to choose another variable from $s_0$ and $t_0$ to form the three independent parameters. $t_0$ can be used to control the stiffness of the manipulator; a smaller $t_0$ can result in larger tension force in the cable which can lead to larger stiffness. Therefore, we consider $s_0$ to be the parasitic motion which can be determined by the other three variables. In fact, this parasitic motion is frequently encountered in parallel manipulators [17].

The homogeneous coordinates of $A_i$ in the fixed base with respect to the frame $OXYZ$ can be described as $Oa_1 = (0, a, 0, 1)^T, Oa_3 = (0, -a, 0, 1)^T, Oa_2 = (-a, 0, 0, 1)^T, Oa_4 = (a, 0, 0, 1)^T$ and corresponding homogeneous coordinates for other ends of the cables $B_i$ in the moving platform with respect to $oxy$ can be described as $ob_1 = (0, b, 0, 1)^T, ob_3 = (0, -b, 0, 1)^T, ob_2 = (-b, 0, 0, 1)^T, ob_4 = (b, 0, 0, 1)^T$. The rotational matrix from the body frame to the fixed frame can be considered as the rotation about an axis $\omega$ perpendicular to $Os$ with an angle $\theta_p$, where $\omega = [-\sin \theta_s, \cos \theta_p, 0]^T$. Then, the homogeneous transformation matrix can be obtained as:

$$
O_T_o = \begin{pmatrix}
   t_{11} & t_{12} & t_{13} & s_0 \cos \theta_s \\
   t_{21} & t_{22} & t_{23} & s_0 \sin \theta_s \\
   t_{31} & t_{32} & t_{33} & t_0 \\
   0 & 0 & 0 & 1
\end{pmatrix}
$$
where

\[ t_{11} = \sin^2 \theta_s + \cos \theta_p \cos^2 \theta_s \]
\[ t_{12} = t_{21} = (\cos \theta_p - 1) \cos \theta_s \sin \theta_s \]
\[ t_{13} = -t_{31} = \sin \theta_p \cos \theta_s \]
\[ t_{22} = \cos^2 \theta_s + \cos \theta_p \sin^2 \theta_s \]
\[ t_{23} = -t_{32} = \sin \theta_p \sin \theta_s \]
\[ t_{33} = \cos \theta_p \]

III. **Combined Inverse Position and Statics Analysis**

For the inverse position analysis, we want to obtain the driven cable length given the desired DOF value. Let \( x = [\theta_s, \theta_p, t_0]^T \in \mathbb{R}^3 \) and \( q = [t_1, t_2, t_3, t_4]^T \in \mathbb{R}^4 \), the inverse kinematics is to find the relation:

\[ q = f(x) \quad f : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \]

If we don’t consider assumption 1, then \( s_0 \) is an independent parameter and should be included in vector \( x \). In this case, \( O_T \) is completely determined, and the inverse position analysis can be solved from \( l_i = |O_T^a b_i - O_T^a a_i| (i = 1, 2, 3, 4) \).

If we consider assumption 1, \( s_0 \) needs to be solved from \( x \). This parasitic motion is a characteristic of the spring bending, which in turn is related to the forces acting on the spring. These forces are mainly resulting from the pulling forces in the four cables. Therefore, we will combine the inverse position analysis and statics in order to obtain a solution.

**A. Force and Moment Balance Equations**

To transform all the forces in four cables to the top center of the spring, we first need to know the four direction vectors \( u_i \) expressed in the fixed coordinate frame by \( O_T^a u_i = \left[ O_T^a a_i - O_T^a T_o b_i \right]/\left| O_T^a a_i - O_T^a T_o b_i \right| \).

With assumption 1, all the forces along the four cables can be transformed into bending plane \( Ost \); otherwise, the spring will not bend in that plane. In other words, we can convert all the forces to two perpendicular forces \( F_1 \) and \( F_2 \) in the plane, and a moment \( M \) perpendicular to the plane at spring’s top center as shown in Fig. 3. Without loss of generality, we assume the moving platform is not subject to external forces. The case when there are external forces such as the weight of the respirators can be analyzed in the same way. The mass of the moving platform is taken as a mass point at spring’s top center with quantity \( m \). For the force and moment balance, we have:

\[
\sum_{i=1}^{4} T_i O_u_i = [0, 0, mg]^T = [F_1 \cos \theta_s, F_1 \sin \theta_s, -F_2]^T (1)
\]

\[
\sum_{i=1}^{4} O_r_i \times T_i O_u_i = [-M \sin \theta_s, M \cos \theta_s, 0]^T (2)
\]

where \( O_r_i \) corresponds to the vector \( \overrightarrow{OB_i} \) expressed in the fixed frame and

\[
O_{r_1} = b[t_{12} + t_{13}, t_{32}]^T \quad O_{r_2} = b[-t_{11}, -t_{21}, -t_{31}]^T
\]
\[
O_{r_3} = b[-t_{12}, -t_{22}, -t_{32}]^T \quad O_{r_4} = b[t_{11}, t_{21}, t_{31}]^T
\]

In Eqs. (1) and (2), there are totally eight unknowns: \( T_i \) to \( T_4 \), \( F_1 \), \( F_2 \), \( M \), and \( s_0 \). We will show that we can eliminate \( T_1 \) to \( T_4 \) and obtain an equation with only \( F_1 \), \( F_2 \), \( M \), and \( s_0 \) as the unknowns. This equation can be combined with another three equations from spring’s lateral bending to solve all the four unknowns. To simplify the notations, let \( F_2' = F_2 - mg \). To eliminate \( T_1 \) to \( T_4 \), we decompose the force and moment balance equations into six equations:

![Fig. 3. The bending plane equivalent force system](image-url)

\[
s_0 \cos \theta_s (T_1' + T_2' + T_3' + T_4') +

(T_2' - T_4')(a - t_{11} b) + (T_1' - T_3')t_{12} b + F_1 \cos \theta_s = 0 (3)
\]

\[
s_0 \sin \theta_s (T_1' + T_2' + T_3' + T_4') -

(T_1' - T_3')(a - t_{22} b) - (T_2' - T_4')t_{12} b + F_1 \sin \theta_s = 0 (4)
\]

\[
b[(T_1' - T_3')t_{12} - (T_2' - T_4')t_{31}] +

t_0(T_1' + T_2' + T_3' + T_4') - F_2' = 0 (5)
\]

\[
(T_2' - T_4')(t_{22} t_{10} - t_{32} s_0 \sin \theta_s) + (T_1' + T_3')a t_{32} +

(T_1' - T_3')(t_{31} \sin \theta_s - t_{21} l_0) - \frac{M}{b} \sin \theta_s = 0 (6)
\]

\[
(T_2' - T_4')(t_{11} t_{10} - t_{31} s_0 \cos \theta_s) - (T_2' + T_4')a t_{31} +

(T_1' - T_3')(t_{32} s_0 \cos \theta_s - t_{12} l_0) + \frac{M}{b} \cos \theta_s = 0 (7)
\]

\[
s_0[(T_1' - T_3') \cos \theta_s + (T_2' - T_4') \sin \theta_s] -
\]

\[
a(\cos \theta_p - 1) \sin \theta_s \cos \theta_s [(T_1' + T_3') - (T_2' + T_4')] = 0 (8)
\]

where \( T_i' = T_i/l_i \). From (3) \times \sin \theta_s - (4) \times \cos \theta_s, we have:

\[
(a - b)(T_1' - T_3') \cos \theta_s + (T_2' - T_4') \sin \theta_s = 0 \quad (9)
\]

From (6) \times \cos \theta_s + (7) \times \sin \theta_s, we have:

\[
t_0[(T_1' - T_3') \cos \theta_s + (T_2' - T_4') \sin \theta_s] -
\]

\[
a \sin \theta_p \sin \theta_s \cos \theta_s [(T_1' + T_3') - (T_2' + T_4')] = 0 \quad (10)
\]
From (8) × \sin \theta_p + (10) × (\cos \theta_p - 1), we have:

\[ [t_0(\cos \theta_p - 1) + s_0 \sin \theta_p][(T'_1 - T'_3) \cos \theta_s + (T'_2 - T'_4) \sin \theta_s] = 0 \]  

(11)

Assume \( a \neq b \) or \( t_0(\cos \theta_p - 1) + s_0 \sin \theta_p \neq 0 \) (the case when \( a = b \) and \( t_0(\cos \theta_p - 1) + s_0 \sin \theta_p = 0 \) will be treated as special case later). Then Eqs. (9) and (11) are of the same form:

\[ (T'_1 - T'_3) \cos \theta_s + (T'_2 - T'_4) \sin \theta_s = 0 \]  

(12)

Plug Eq. (12) into Eq. (10). Since \( a \neq 0 \) and \( \sin \theta_p \neq 0 \) for small \( \theta_p \), we have

\[ T'_2 + T'_4 = T'_1 + T'_3 \]  

(13)

provided \( \theta_s \neq \frac{k\pi}{2} \), \( k = 0, 1, 2, \ldots \) (the case when \( \theta_s = \frac{k\pi}{2} \) will be treated as special case later).

\[ \begin{align*}
T'_1 - T'_3 & = \frac{\sin \theta_s(t_0F_1 + s_0F'_2)}{t_0(a - b \cos \theta_p) - b s_0 \sin \theta_p} \\
T'_1 + T'_3 & = \frac{F_1 b \sin \theta_p + F'_2(a - b \cos \theta_p)}{2t_0(a - b \cos \theta_p) - 2b s_0 \sin \theta_p}
\end{align*} \]  

(14)

With Eqs. (12), (13), (6), (14), and (15), we can eliminate \( T'_1 \) to \( T'_3 \) and have:

\[ 2bF'_2 \sin \theta_p s_0^2 + 2b(t_0 F_1 \sin \theta_p + M \sin \theta_p + t_0 F'_2 \cos \theta_p) s_0 + (2b t_0 \cos \theta_p - ab^2 \sin^2 \theta_p) F_1 - ab \sin \theta_p (a - b \cos \theta_p) F'_2 - 2t_0(a - b \cos \theta_p) M = 0 \]  

(16)

In this equation, the unknown variables are \( F_1, F'_2, M, s_0 \). In the next subsection, we will show that \( F'_2 \) can be obtained through \( t_0 \), while \( F_1 \) and \( M \) can be expressed as a linear function of \( s_0 \) from the lateral bending of the spring. In this way, we can solve Eq. (16) for \( s_0 \) and all the other unknown variables can be obtained.

### B. Lateral Bending Equations for the Spring

Assume the notations of the selected compression helical spring as following: \( K \) is the spring constant; \( l_0 \) is the initial length of the spring; \( h_0 \) is the pitch of the spring; \( n \) is the number of coils; \( r \) is the radius of the spring; \( l_0 \) is the length of the spring after compression; \( \beta_0 \) is the flexural rigidity of the spring and corresponds to the same physical meaning for the solid bar; \( \beta \) is the same quantity after compression of the spring. For \( \beta \) and \( \beta_0 \), we have:

\[ \beta = \beta_0 \frac{l_0}{l_0} \quad \text{and} \quad \beta_0 = \frac{2EGh_0}{\pi r(E + 2G)} \]

where \( E \) is the elastic modulus, \( G \) is the shearing modulus, and \( I \) is the moment of inertia of the cross section of the wire with respect to its diameter [18].

Since after compression, the spring length is \( l_0 \) and the compression length is usually small, we can approximate \( F_2 \) as:

\[ F_2 = K(l_0 - t_0) \]  

(17)

Consider the practical lateral bending of the neck in Fig. 3. The spring-based mechanism will be bent by a force \( F_1 \) and \( F'_2 \) plus a couple \( M \). For any cross section of the spring, the following equation is satisfied for small deflection:

\[ \beta \frac{d^2 s}{dt^2} = M + F_2(s_0 - s) + F_1(t_0 - t) \]  

(18)

The solution for this equation is:

\[ s = C_1 \cos(\sqrt{F_2/\beta}t) + C_2 \sin(\sqrt{F_2/\beta}t) - \frac{F_1}{F_2} t + \frac{M + F_2s_0 + F_1t_0}{F_2} \]  

(19)

where \( C_1 = -(M + F_2s_0 + F_1t_0)/F_2 \) and \( C_2 = F_1\sqrt{\beta}/F_2^2 \) are constants obtained from two initial conditions for built-in end:

\[ s(t = 0) = s_0 \quad s'(t = 0) = \tan \theta_p \]

Plugging the other two initial conditions for the free end:

\[ s(t = t_0) = s_0 \quad s'(t = t_0) = \tan \theta_p \]

to Eq. (19), we can obtain

\[ \begin{align*}
a_1 F_1 + b_1 F_1 + c_1 s_0 + d_1 & = 0 \\
a_2 F_1 + b_2 F_1 + c_2 s_0 + d_2 & = 0
\end{align*} \]  

(20)

(21)

where

\[ \begin{align*}
a_1 & = 1 - \cos(\sqrt{F_2/\beta}t_0) \quad a_2 = \sqrt{F_2/\beta} \sin(\sqrt{F_2/\beta}t_0) \\
b_1 & = \sqrt{\beta/F_2} \sin(\sqrt{F_2/\beta}t_0) - t_0 \cos(\sqrt{F_2/\beta}t_0) \\
b_2 & = \sqrt{\beta/F_2} \sin(\sqrt{F_2/\beta}t_0) + t_0 \sqrt{F_2/\beta} \sin(\sqrt{F_2/\beta}t_0) - 1 \\
c_1 & = -F_2 \cos(\sqrt{F_2/\beta}t_0) \quad c_2 = F_2 \sqrt{\beta/F_2} \sin(\sqrt{F_2/\beta}t_0)
\end{align*} \]

\[ \begin{align*}
d_1 & = 0 \\
d_2 & = -F_2 \tan \theta_p
\end{align*} \]

These two equations are linear equations of \( M, F_1 \), and \( s_0 \), where all the coefficients are known. Therefore, from them we can express \( M \) and \( F_1 \) as a function of \( s_0 \):

\[ F_1 = D_1 s_0 + E_1 \]  

(22)

\[ M = D_2 s_0 + E_2 \]  

(23)

where

\[ \begin{align*}
D_1 & = -\frac{a_2 c_1 - a_1 c_2}{a_2 b_1 - a_1 b_2} \quad E_1 = -\frac{a_2 d_1 - a_1 d_2}{a_2 b_1 - a_1 b_2} \\
D_2 & = -\frac{b_2 c_1 - b_1 c_2}{b_2 a_1 - b_1 a_2} \quad E_2 = -\frac{b_2 d_1 - b_1 d_2}{b_2 a_1 - b_1 a_2}
\end{align*} \]

Plug Eqs. (22) and (23) to Eq. (16), we can get a quadratic equation of \( s_0 \):

\[ A s_0^2 + B s_0 + C = 0 \]  

(24)

where

\[ \begin{align*}
A & = 2b \sin \theta_p (F'_2 + t_0 D_1 + D_2) \\
B & = 2b(t_0(F'_2 \cos \theta_p + E_1 \sin \theta_p) + 2b E_2 \sin \theta_p + (2b t_0 \cos \theta_p - ab^2 \sin^2 \theta_p) D_1 - 2t_0(a - b \cos \theta_p) D_2 \\
C & = (2b t_0 \cos \theta_p - ab^2 \sin^2 \theta_p) E_1 - ab \sin \theta_p(a - b \cos \theta_p) F'_2 - 2t_0(a - b \cos \theta_p) E_2
\end{align*} \]
Denote the set of nonnegative real numbers as $\mathbb{R}_{\geq 0}$, then we need to find $s_0 \in \mathbb{R}_{\geq 0}$. Once $s_0$ is obtained, all the other unknowns can be solved. Note that we have nine equations, Eqs. (3) to (8), (17), (20), and (21), to solve eight unknowns ($T_1$ to $T_6$, $F_1$, $F_2$, $M$, and $s_0$). Generally there is no solution; however, since Eqs. (9) and (11) are the same if the conditions stated before are hold, there are only eight equations for eight unknowns.

C. Special Cases

In above derivation, we have made the assumptions $a \neq b$ or $t_0(\cos \theta_p - 1) + s_0 \sin \theta_p \neq 0$ to obtain Eq. (12), and $\theta_s \neq \frac{k\pi}{2}$, $k = 0, 1, 2, \ldots$ to obtain Eq. (13). In this subsection, we will discuss these possibilities.

1) $(T_1' - T_3')\cos \theta_s + (T_2' - T_4')\sin \theta_s \neq 0$: If $(T_1' - T_3')\cos \theta_s + (T_2' - T_4')\sin \theta_s \neq 0$, we have both $a = b$ and $t_0(\cos \theta_p - 1) + s_0 \sin \theta_p = 0$ with the latter condition can be simplified as $s_0/t_0 = \tan(\theta_p/2)$. However, we will show that if $a = b$, we will have $s_0/t_0 \geq \tan(\theta_p/2)$ where the equality only happens when $\theta_p = 0$. This means we will always have $(T_1' - T_3')\cos \theta_s + (T_2' - T_4')\sin \theta_s = 0$ provided $\theta_p \neq 0$.

In fact, under the condition of $a = b$, the three decomposed equations for Eq. (1) are:

$$\cos \theta_s [s_0(T_1' + T_2' + T_3' + T_4') + a(1 - \cos \theta_p)] + \sin \theta_s [s_0(T_2' + T_3' + T_4' + T_1') + a(1 - \cos \theta_p)] = 0 \quad (25)$$

$$\cos \theta_s [s_0(T_2' - T_4')\cos \theta_s - (T_2' - T_4')\sin \theta_s + F_1] = 0 \quad (26)$$

$$\cos \theta_s [s_0(T_2' - T_4')\cos \theta_s - (T_2' - T_4')\sin \theta_s + F_1] = 0 \quad (27)$$

Note that Eqs. (25) and (26) are the same if $\theta_s \neq \frac{k\pi}{2}$.

If $\frac{s_0}{t_0} = \tan \frac{\theta_p}{2}$, then from the above three equations we have $F_1t_0 + F_2s_0 = 0$ which, combined with Eqs. (20) and (21), can be used to obtain a solution of $s_0$:

$$s_0' = \frac{a_2d_2 - a_1d_1}{a_2(c_1 - \frac{b_1F_2}{t_0}) - a_1(c_2 - \frac{b_2F_2}{t_0})}$$

We expect that $s_0' = s_0$ so that $s_0'/t_0 = s_0/t_0$; however, this is only valid when $\theta_p = 0$. An analytical derivation is complicated, but we can use the numerical approach to see the relation. Because we will assume $t_0$ is fixed in the following subsection, we plot the 3D graph using the $\theta_s$ and $\theta_p$ as the $x$ and $y$ coordinates and $s_0'/t_0 - s_0/t_0$ as the $z$ coordinate. The figure for $t_0 = 0.085m$, $a = b = 0.05m$, $\theta_s \in [0, 2\pi]$, and $\theta_p \in [0, \pi/2)$ is shown as Fig. 4. The parameters of compression spring we are using for the prototype are $l_0 = 0.1016m$, $h_0 = 0.0195m$, $G = 81.2GPa$, $E = 196.5GPa$, $r = 22.73mm$, $K = 4153N/m$, and the diameter of the spring wire $d = 3.76mm$. Therefore, we can get $\beta_0 = 0.2321$. From the figure, we see that $s_0'/t_0 - s_0/t_0$ is strictly monotonically increasing with respect to $\theta_p$, and the only case for $s_0'/t_0 = s_0/t_0$ is when $\theta_p = 0$. The case for $\theta_p = 0$ is not interested as there is no bending motion of the platform. We will assume $\theta_p \neq 0$ in the following to avoid this special case and to ensure $(T_1' - T_3')\cos \theta_s + (T_2' - T_4')\sin \theta_s \neq 0$ is always hold.

2) $\theta_s = \frac{k\pi}{2}$, $k = 0, 1, 2, \ldots$: For Eq. (13), without loss of generality, suppose $\theta_s \in [0, 2\pi)$, then when $a \neq b$, there exist four special cases corresponding to $\theta_s = 0, \theta_s = \pi/2, \theta_s = \pi$, and $\theta_s = 3\pi/2$. We give the detail derivation for $\theta_s = 0$, and for the other three cases, the procedure is similar.

When $\theta_s = 0$, Eqs. (4), (6), and (8) all degenerate to $T_1' = T_3'$, and the other three equations for force and moment balance becomes:

$$s_0(T_1' + T_2' + T_3' + T_4') + (T_2' - T_4')(a - b\cos \theta_p) + F_1 = 0$$

$$t_0(T_1' + T_2' + T_3' + T_4') + (T_2' - T_4')b\sin \theta_p - F_2 = 0$$

$$a\sin \theta_p(T_2' + T_4') + (T_2' - T_4')(t_0 \cos \theta_p + s_0 \sin \theta_p) + \frac{M}{b} = 0$$

Combining the other three equations from spring bending, we can not get a unique solution because there are only seven equations for eight unknowns. The reason is that we can have infinitely many values for $T_1'$ and $T_3'$ provided that they are equal. Therefore, we can force the value to satisfy the following equation

$$T_1' + T_3' = T_2' + T_4'$$

to make it compatible to the analysis for non-special case.

In this case, eliminating $T_1', T_2', T_3'$, and $T_4'$, we can get the same equation as Eq. (16). Therefore, the above procedure can also be used to derive the solution for such four special cases.

IV. NumerICAL IMPLEMENTATIONS

The position and statics analysis is implemented in Matlab$^\text{TM}$. The parameters are chosen as $a = 0.06m$, $b = 0.05m$, $m = 0.05kg$. The implementation is performed with a fixed $l_0 = 0.085m$ because in real application, $l_0$ can be used to adjust the pretension force in the four cables. By varying $\theta_p$ from 0 to $\pi/9$ and $\theta_s$ from 0 to $2\pi$, we can obtain the results shown in Fig. 5, where the cable length or force are shown as the $z$ coordinate.
We can have three observations from the figure. First of all, the length and force are complement to each other for each cable. In other words, when the length is small, the force will be large. This is in accordance with our intuition that when the cable length is small, the cable should exert large force on the moving platform. The second characteristic is that when θ_p is large, the variations of both the force and length are also large. This is because the more we want to make the moving platform tilt, the larger force we need to exert on it. Finally, for a fixed θ_p, the length and force curve for all the cables with θ_s from 0 to 2π is symmetric.

V. CONCLUSIONS

This paper gives a general solution to the kinematics and statics for a cable driven parallel manipulator with a spring as the spine. Different from traditional parallel manipulator, we show that the inverse position and statics should be solved simultaneously because of the parasitic motion of the spring. The parasitic motion is obtained using spring bending equations. Numerical implementations verify the correctness of proposed mathematical model. The analysis performed in this paper can be applied to other parallel manipulators with flexible spines.

REFERENCES