Taylor Series and Laurent Series:

Before considering these power series we re-examine the behavior of an analytic function inside a region R enclosed by the contour C. The Cauchy Integral Formula provides the identity

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

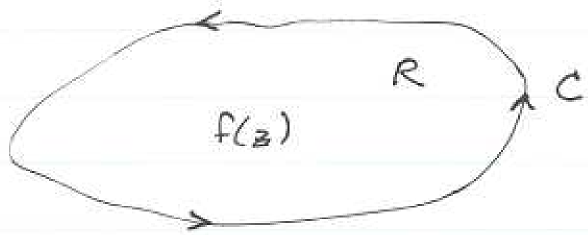
Then: $$\frac{f(\xi)-f(z)}{\xi - z} = \frac{1}{2\pi i}\left[\oint_C \left(\frac{f(\zeta)}{\zeta-\xi} - \frac{f(\zeta)}{\zeta - z}\right)\frac{d\zeta}{\xi - z}\right] = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\xi - z}\left[\frac{\cancel{\zeta} - z - (\cancel{\zeta} - \xi)}{(\zeta-\xi)(\zeta - z)}\right] d\zeta$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-\xi)(\zeta-z)}\, d\zeta$$

Hence: $$\frac{df}{dz} = f'(z) = \lim_{\xi \to z}\frac{f(\xi)-f(z)}{\xi - z} = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

Continuing: $$f''(z) = \frac{2}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta, \ldots, \quad f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

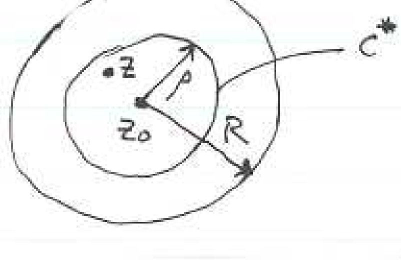
A function $f(z)$ that is analytic inside region R enclosed by contour C has derivatives of all orders in R



We can now formulate Taylor's and Laurent's series for complex functions.

<u>Taylor's Series:</u> Function $f(z)$, analytic inside a circle with $|z - z_0| < R$, can be represented in the circle as a convergent power series,

$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, c_n = unique.



$|z - z_0| < R$

Radius $\rho < R$ centered at z_0. Contains z

Now z is an interior point of the circle of radius ρ, i.e., $|z-z_0|<\rho$. Function $f(z)$ is analytic in that circle, therefore

$$f(z)=\frac{1}{2\pi i}\oint_{C^*}\frac{f(s)}{s-z}\,ds$$

Write $\displaystyle \frac{1}{s-z}=\frac{1}{(s-z_0)\left[1+\frac{z_0-z}{s-z_0}\right]}=\frac{1}{s-z_0}\,\frac{1}{\left[1-\frac{z-z_0}{s-z_0}\right]}=\frac{1}{s-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^n,$

which we can write because $\frac{z-z_0}{s-z_0}$ has magnitude less than unity because z is an interior point while s is a boundary point of C^*.
Hence, we have

$$f(z)=\frac{1}{2\pi i}\oint\frac{f(s)}{s-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^n=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C^*}\frac{f(s)}{(s-z_0)^{n+1}}\,ds$$

$$=\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!}\,\frac{n!}{2\pi i}\oint_{C^*}\frac{f(s)}{(s-z_0)^{n+1}}\,ds=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$$

$$=\sum_{n=0}^{\infty}c_n(z-z_0)^n$$

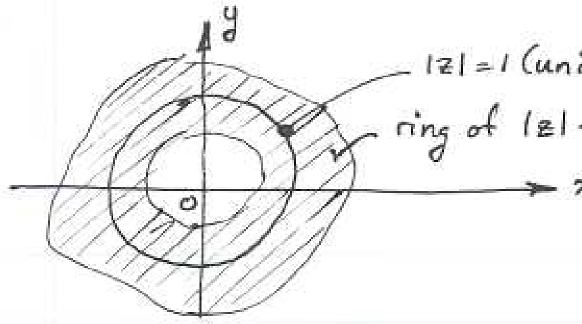
with $\displaystyle c_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi i}\oint_{C^*}\frac{f(s)}{(s-z_0)^{n+1}}\,ds.$

<u>Example:</u> $f(z)=\ln z$ with $z_0=1$. Since $\ln z=\int_1^z\frac{ds}{s}$ we have
$f'(z)=1/z$, $f''(z)=-1/z^2$, $f'''(z)=2!/z^3$, $f^{IV}=-3!/z^4$, ...
Using $z_0=1$ gives: $f(z_0)=f(1)=\ln 1=0$; $f'(z_0)=f'(1)=1$; $f''(1)=-1$; $f'''(1)=2!$;
$f^{IV}(1)=-3!$ etc. Thus: $c_0=f(z_0)=0$; $c_1=f'(z_0)/1!=1$; $c_2=f''(z_0)/2!=-\frac{1}{2}$; $c_3=f'''(z_0)/3!$
$=2!/3!=1/3$; $c_4=f^{IV}(z_0)/4!=-3!/4!=-1/4$; etc...

$$\ln z=\sum_{n=0}^{\infty}c_n(z-z_0)^n=\sum_{n=0}^{\infty}c_n(z-1)^n=0+(z-1)-\tfrac{1}{2}(z-1)^2+\tfrac{1}{3}(z-1)^3-\tfrac{1}{4}(z-1)^4+\dots$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n$$

Near $z=1$, $\ln z$ is an analytic, infinitely differentiable function. The "point" $z=1$ consists of an infinite number of point-pairs (x,y) since $z=x+iy=\sqrt{x^2+y^2}\,e^{i\tan^{-1}y/x}$ and $z\approx 1$ means $\sqrt{x^2+y^2}\approx 1$. In the (x,y) plane $z\sim 1$ is a ring about the unit circle perimeter:



$|z|=1$ (unit circle)

ring of $|z|\approx 1$: Here, $\ln z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n$

Laurent Series: The preceding discussion does not exclude $n \leq 0$. Let us generalize to the case where $f(z)$ has poles:

$$f(z) = \sum_{n=-\infty}^{n=\infty} c_n (z-z_0)^n \; ; \quad c_n = \frac{1}{2\pi i} \oint_{C^*} \frac{f(s)}{(s-z_0)^{n+1}} \, ds$$

The point $z = z_0$ is called an isolated singularity. Now define

Residue: The "residue" of $f(z)$ at the isolated singular point z_0 is defined as

$$c_{-1} = \frac{1}{2\pi i} \oint_{C^*} f(s)\, ds$$

Note: If near $z = z_0$ we can write $f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots$ then

$$c_{-1} = \lim_{z \to z_0} (z-z_0) f(z).$$

Note: Write $f(z)$ as the quotient of $\phi(z)$, $\psi(z)$, i.e., $f(z) = \phi(z)/\psi(z)$. Near $z = z_0$ suppose $\phi(z_0) \neq 0$ and $\psi(z) = \psi(z_0) + (z-z_0)\psi'(z_0) + \cdots$ with $\psi(z_0) = 0$. Then near $z = z_0$

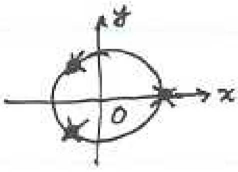
$$f(z) = \frac{\phi(z)}{\left[(z-z_0)\psi'(z_0) + \frac{(z-z_0)^2}{2!}\psi''(z_0) + \cdots\right]}$$

Then the residue

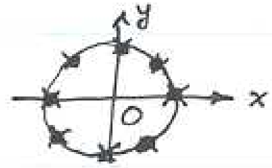
$$c_{-1} = \lim_{z \to z_0} (z-z_0) f(z) = \lim_{z \to z_0} \frac{\phi(z)}{\psi'(z_0) + \frac{(z-z_0)}{2}\psi''(z_0) + \cdots} = \frac{\phi(z_0)}{\psi'(z_0)}.$$

<u>Example:</u> $f(z) = \dfrac{z}{z^n - 1}$. Here $\phi(z) = z$, $\psi(z) = z^n - 1$ and $\psi(z)$ is singular at $z_k = \sqrt[n]{1} = e^{i(2k\pi/n)}$ with $k = 0, 1, 2, \ldots, n-1$

$k = 3$:



$k = m$ (8)



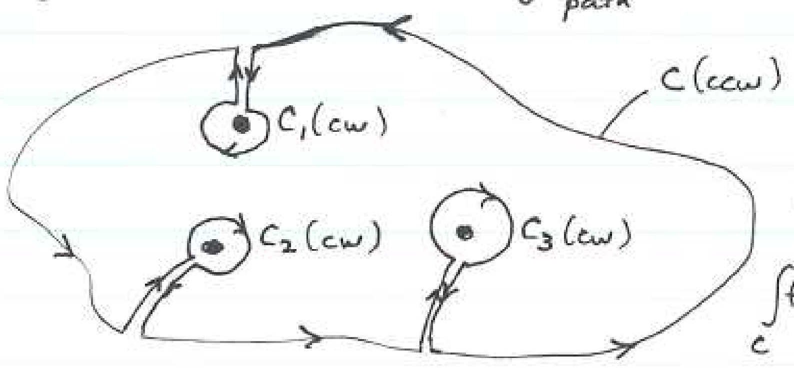
These singular points are all poles.

$$C_{-1} = \frac{\phi(z_0)}{\psi'(z_0)} \Rightarrow \frac{z_k}{n z_k^{n-1}} = \frac{e^{i(2k\pi/n)}}{n e^{i\left(2k\pi\frac{(n-1)}{n}\right)}} = \frac{e^{i(2k\pi/n)}}{n e^{i(2k\pi - 2k\pi/n)}} = \frac{e^{i\left(\frac{4k\pi}{n}\right)}}{n}$$

This function has n residues.

We now have sufficient tools to combine the notions of analytic function, the Cauchy formula, and the idea of residues into powerful tools and techniques for evaluating integrals in the complex plane. Often the domain has a segment over which the integral is real so the complex evaluation can be used to derive real results.

Consider an analytic function $f(z)$, with singularities (poles) at various points. We define an integration contour that <u>excludes</u> these poles, as shown below. Along this integration contour the Cauchy formula gives simply $\oint_{path} f(z)\,dz = 0$.



$$\int_{C + C_1 + C_2 + \cdots} f(z)\,dz = 0$$

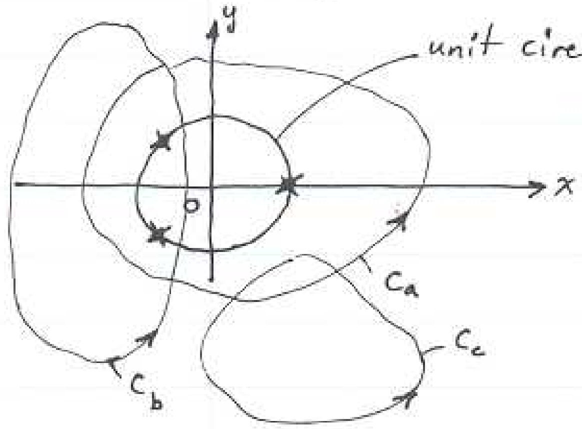
Then:

$$\int_C f(z)\,dz - \int_{-C_1} f(z)\,dz - \ldots - \int_{-C_n} f(z)\,dz = 0$$

But around the poles each of the $-C_i$ integrals produces $2\pi i B_i$, where $B_i = (C_{-1})_i$ is the residue of the i^{th} point. **Thus:**

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} B_j = 2\pi i \sum_{j=1}^{n} (\text{residue})_j$$

Example: $\oint_C \frac{z}{z^3-1}\,dz$. The value of this integral depends on C.



The roots of $z^3 = 1$ are $e^{i(2k\pi/3)}$, $k = 0, 1, 2$ (as shown in the figure).

On C_a: $$\oint_{C_a} \frac{z}{z^3-1}\,dz = 2\pi i \sum_{k=0}^{2} (\text{residues})_k$$

$$= 2\pi i \sum_{k=0}^{2} \frac{e^{i(4k\pi/3)}}{3}$$

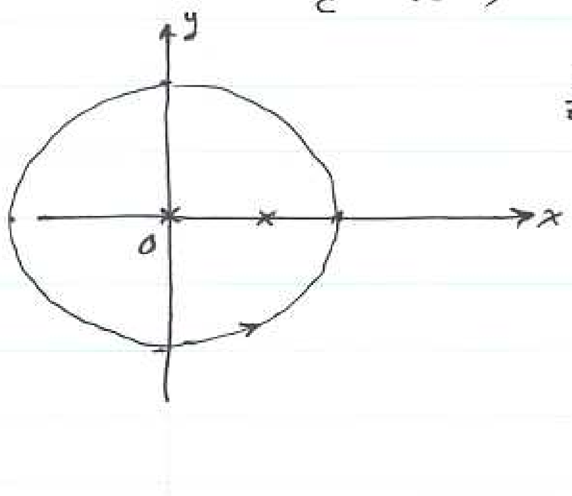
$$= \frac{2\pi i}{3}\left[e^0 + e^{i\frac{4\pi}{3}} + e^{i\frac{8\pi}{3}}\right] = \frac{2\pi i}{3}\left[1 - 2\cos\pi/3\right]$$

On C_b: $$\oint_{C_b} \frac{z}{z^3-1}\,dz = 2\pi i \sum_{k=1}^{2} (\text{residues})_k = \frac{2\pi i}{3}\left[e^{i4\pi/3} + e^{i8\pi/3}\right] = \frac{2\pi i}{3}\left[-2\cos\pi/3\right]$$

On C_c: $$\oint_{C_c} \frac{z}{z^3-1}\,dz = 2\pi i \sum_{k=0} (\text{residues})_k = 0$$ (this contour encloses no poles).

The moral: If contour does not enclose a pole it makes NO CONTRIBUTION.

Example: $I = \oint_C \frac{5z-2}{z(z-1)}$ on the circle C with $|z| = 2$.



$$\frac{5z-2}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{2}{z} + \frac{3}{z-1}$$

$$I = \oint_C \frac{2}{z}\,dz + \oint_C \frac{3}{z-1}\,dz = 2\pi i(2) + 2\pi i(3) = 10\pi i$$

$C_{-1} = \lim_{z\to 0} z \cdot \frac{2}{z} = 2 \qquad C_{-1} = \lim_{z\to 1} (z-1)\frac{3}{z-1} = 3$

Note: If $f(z)$ has an isolated singularity, then we write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{in } 0 < |z-z_0| < r.$$

The $(z-z_0)^{-n}$ terms represent the "principal part of $f(z)$ at z_0."

There are three cases to consider:

1. $b_m \neq 0$ with $b_{m+1} = b_{m+2} = \dots = 0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}.$$

Here z_0 is an "isolated singularity of order m," or a pole of order m.
When $m=1$, z_0 = simple pole. When $m=2$, z_0 = pole of order 2.
The residue at z_0 is b_1.

Example: $f(z) = \dfrac{z^3 - 2z + 3}{z-2} = 2 + \dfrac{3}{z-2}$

$z=2$ is a simple pole.
Residue of $f(z)$ at $z=2$ is 3.

Example: $f(z) = \dfrac{\sinh z}{z^4} = \dfrac{1}{z^4}\left[z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \dfrac{z^7}{7!} + \dots\right] = \dfrac{1}{z^3} + \dfrac{1/6}{z} + \dfrac{z}{5!} + \dots$

$z=0$ is a pole of order 3.

Residue at $z=0$ is $1/6$.

2. Infinite number of b_m's $\Rightarrow$ z_0 is an essential singularity.

Example: $e^{1/z} = \displaystyle\sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$

$z=0$ is an essential singular point.

Residue at $z=0$ is 1.

3. All coefficients b_n at z_0 are zero $\Rightarrow$ z_0 is a removable singularity.

Example: $f(z) = \dfrac{\sinh z}{z} = \dfrac{1}{z}\left(z + \dfrac{z^3}{3!} + \dots\right) = 1 + \dfrac{z^2}{3!} + \dots$