

Summation of Series:

Harmonic series: $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{i} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$

Nicole d'Oresme (circa 1323-1382): $1 = \frac{1}{2} + \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{2} \quad \left(\frac{1}{3} + \frac{1}{4} = \frac{7}{12} \right)$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2} \quad \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = 0.634524 > \frac{1}{2} \right)$$

Thus:

$$S = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4} \right)}_{\text{bigger than } \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right)}_{\text{bigger than } \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \right)}_{\text{bigger than } \frac{1}{2}} + \dots$$

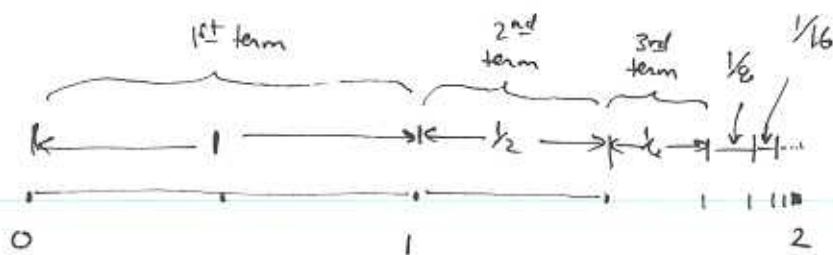
By taking two terms ($\frac{1}{3}, \frac{1}{4}$), then four terms ($\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$), then eight, sixteen, thirty-two, etc., one can ^(arrange) group this series into an infinite group of blocks, each of which is greater than $\frac{1}{2}$. The more of these groups are included, the bigger is the value of S . Hence, S is unbounded. The entire sum, therefore, diverges (it can always be made larger than any chosen #!).

Geometric series: $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{n=0}^{\infty} \frac{1}{2^n}$

This series converges. Consider the partial (finite) sum $S_i = \sum_{n=0}^i \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^i}$. Then $S_i - 1 = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{i-1}} \right) = \frac{1}{2} S_{i-1} = \frac{1}{2} \left(S_i - \frac{1}{2^i} \right)$.

Solving for S_i : $S_i = \frac{1 - \frac{1}{2^{i+1}}}{1 - \frac{1}{2}}$. As i becomes very large, $\frac{1}{2^{i+1}}$ becomes very small (e.g., when $i = 99$ have $\frac{1}{2^{100}} = 7.9 \times 10^{-31}$). Thus,
 $\lim_{i \rightarrow \infty} S_i = S_{\infty} = \frac{1}{1 - \frac{1}{2}} = 2$.

This geometric series converges to the value 2. This convergence can be shown by examining the series graphically, term-by-term.



Note: We can replace the $\frac{1}{2}$ by any number with magnitude less than unity, obtaining $S = \sum_{n=0}^{\infty} x^n$ ($x = \frac{1}{2}$ in previous example).

Using the same approach:

$$S_i = \frac{1 - x^{i+1}}{1 - x} \rightarrow \frac{1}{1 - x} \text{ as } i \rightarrow \infty \text{ (since } x^{i+1} \rightarrow 0 \text{)}$$

x	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1
S	2	$\frac{3}{2}$	$\frac{4}{3}$	1	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$ (?)

Put $x = -1$ in the series to get $S = 1 - 1 + 1 - 1 + 1 - 1 + \dots$
 Stop at an even number and $S = 0$. Stop at an odd number and $S = 1$.
 Therefore: NO VALUE!

Oscillatory series:

$$\begin{aligned}
 S &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \dots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \\
 &= \left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right) - \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \dots\right) \\
 &= \left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right) - \frac{1}{2} \left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right) \\
 &= S_0 - \frac{1}{2} S_0 = \frac{S_0}{2},
 \end{aligned}$$

where: $S_0 = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots = 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$

This is the geometric series with $x = \frac{1}{4}$ whose value (sum) is $\frac{4}{3}$.

Thus:

$$S = \frac{\frac{4}{3}}{2} = \frac{4}{6} = \frac{2}{3}.$$

A precursor to the Riemann Zeta Function: Consider the sum

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

This series has the sum $S_2 = \pi^2/6$. Similarly, $S_4 = \sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \pi^4/90$ and $S_6 = \sum_{n=1}^{\infty} \frac{1}{n^6} = \pi^6/945$.

Nobody has yet developed a closed-form solution for the series $S_3 = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots = 1.202056903159\dots$

(this answer was obtained by brute calculation). In 1978 Roger Apéry proved this sum was an irrational number. The number $1.2020569031595942854\dots$ is called "Apéry's number."

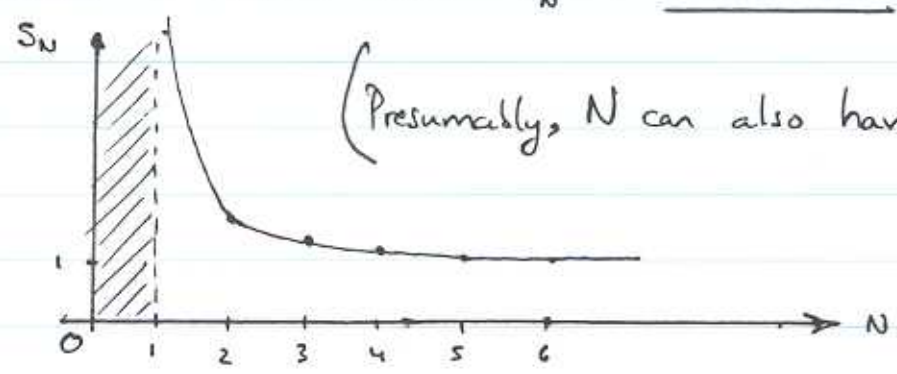
Similarly, no closed-form solutions exist for $\sum_{n=1}^{\infty} \frac{1}{n^5}$, $\sum_{n=1}^{\infty} \frac{1}{n^7}$, etc., but brute-force sums can be found. Convergence (ratio test) can be proved.

Consider: $S_N = 1 + \frac{1}{2^N} + \frac{1}{3^N} + \frac{1}{4^N} + \frac{1}{5^N} + \dots$

Table:

N	1	2	3	4	5	6
S_N	∞	$\pi^2/6 = 1.644934\dots$	$1.202056\dots$	$\pi^4/90 = 1.08232\dots$	$1.0369\dots$	$1.017\dots$

As N increases it seems that $S_N \rightarrow 1$ from above.



(Presumably, N can also have fractional powers.)

Consider once again the case S_2 and write

$$S_2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

Write

$$\frac{1}{2^2} S_2 = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$$

$$\text{Thus: } S_2 - \frac{1}{2^2} S_2 = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = S_2 \left(1 - \frac{1}{2^2}\right)$$

Now write:

$$\frac{1}{3^2} \left(S_2 - \frac{S_2}{2^2}\right) = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \dots$$

Subtract this expression from $S_2 \left(1 - \frac{1}{2^2}\right)$ to get:

$$S_2 \left(1 - \frac{1}{2^2}\right) - \frac{1}{3^2} S_2 \left(1 - \frac{1}{2^2}\right) = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{23^2} + \frac{1}{25^2} + \dots$$

$$S_2 \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{23^2} + \frac{1}{25^2} + \dots$$

Repeat the same process, multiplying this last series by $\frac{1}{5^2}$ and subtracting it from the last series:

$$S_2 \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{5^2}\right) = 1 + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{23^2} + \frac{1}{29^2} + \dots$$

Clearly, the more we do this the smaller the terms on the RHS become, and eventually the RHS $\rightarrow 1$. Thus,

$$S_2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{1}{\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{5^2}\right) \left(1 - \frac{1}{7^2}\right) \left(1 - \frac{1}{11^2}\right) \dots}$$

Some remarkable facts:

- 1) The infinite sum on the LHS is equal to an infinite product on the RHS.

2) The infinite product contains only prime numbers 2, 3, 5, 7, 11, 13, We write:

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_p (1 - p^{-2})^{-1} \quad \text{where } p = 2, 3, 5, 7, 11, \dots \text{ are the primes.}$$

The Riemann Zeta Function: We simply generalize the series of the previous sub-section and write

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

When $s=1$ we have $\zeta(1)=\infty$, when $s=2$ $\zeta(2)=\pi^2/6$ and so on. The development of the previous sub-section also follows nearly verbatim, giving

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}.$$

Note: This expression indirectly proves that the number of primes is infinite. Consider $s=1$, and that there are only a finite number of primes. Then the RHS is finite. The LHS, however, is infinite. This is illogical, therefore the number of primes must be infinite so the RHS $\prod_p (1 - p^{-1})^{-1}$ also diverges.

The function $\zeta(s)$ becomes Riemann's Zeta Function when one takes s to be a complex number, $s = \sigma_r + i\sigma_i$.