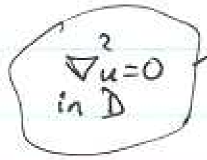
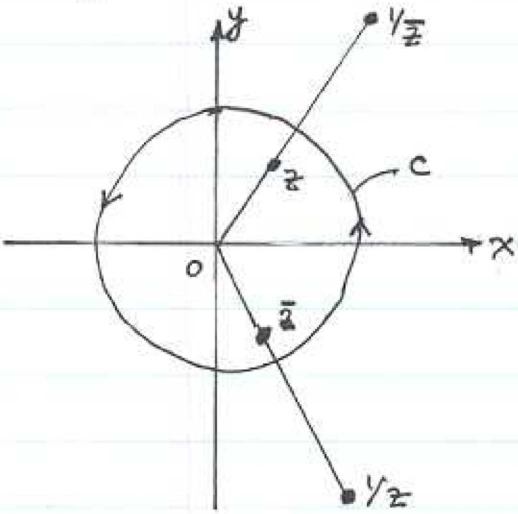
Harmonic Functions: The maximum/minimum principle:

If $u(x,y)$ is harmonic in domain D inculding boundary (∂D) then $u(x,y)$ takes its max. and min. values on ∂D.



Uniqueness: $u_1(x,y)$ and $u_2(x,y)$ satisfy $\nabla^2 u_i = 0$, and have same boundary values. Then let $\phi = u_1 - u_2$ so $\nabla^2 \phi = 0$ with $\phi = 0$ on boundary. Zero is both the max. and min. value so $\phi = 0$ everywhere. QED.

An analytic function is therefore determined uniquely from its boundary values as shown by Cauchy's Integral Formula. With no loss of generality, let boundary C be the unit circle centered on the origin of coordinates.



$$2\pi i\, f(z) = \oint_{\substack{\text{unit}\\ \text{circle}}} \frac{f(\zeta)\, d\zeta}{\zeta - z} \quad ; \quad |z| \le 1$$

Let $\zeta = e^{i\theta} \Rightarrow d\zeta = e^{i\theta}\, i\, d\theta = \zeta\, i\, d\theta$

$$2\pi \cancel{i} f(z) = \int_{\theta=0}^{2\pi} \frac{f(\zeta)}{\zeta - z}\, \zeta \cancel{i}\, d\theta$$

From the figure we note that if z lies inside the unit circle then $1/z$ and $1/\bar{z}$ are outside the unit circle. Then $\int_0^{2\pi} \frac{f(\zeta)}{\zeta - 1/\bar{z}}\, \zeta\, d\theta = 0$.

both work!

$$\text{Thus:} \quad 2\pi f(z) = \int_{\theta=0}^{2\pi} f(\zeta)\left[\frac{\zeta}{\zeta - z} \pm \frac{\zeta}{\zeta - 1/\bar{z}}\right] d\theta \qquad \text{(a)}$$

Write: $\dfrac{\zeta}{\zeta - 1/\bar{z}} = \dfrac{\zeta\bar{\zeta}}{\zeta\bar{\zeta} - \bar{\zeta}/\bar{z}} = \dfrac{1}{1 - \bar{\zeta}/\bar{z}} = \dfrac{\bar{z}}{\bar{z} - \bar{\zeta}}$, since $\zeta = e^{i\theta}$ and $\bar{\zeta} = e^{-i\theta}$.

Then: $\dfrac{\zeta}{\zeta - z} - \dfrac{\zeta}{\zeta - 1/\bar{z}} = \dfrac{\zeta}{\zeta - z} - \dfrac{\bar{z}}{\bar{z} - \bar{\zeta}} = \dfrac{\zeta(\cancel{\bar{z}} - \bar{\zeta}) - \bar{z}(\cancel{\zeta} - z)}{(\zeta - z)(\bar{z} - \bar{\zeta})} = \dfrac{-1 + |z|^2}{-|\zeta - z|^2} = \dfrac{1 - |z|^2}{|\zeta - z|^2}$: REAL

Use this in (a), with $f(\zeta) = f(e^{i\theta}) = u(\theta) + iv(\theta)$.

$f(z) = f(re^{i\phi}) = u(r,\phi) + iv(r,\phi) \qquad (z = re^{i\phi})$.

$$2\pi\left(u(r,\phi) + iv(r,\phi)\right) = \int_0^{2\pi} u(\theta)\, \frac{1-|z|^2}{|\zeta - z|^2}\, d\theta + i \int_0^{2\pi} v(\theta)\, \frac{1-|z|^2}{|\zeta - z|^2}\, d\theta. \quad \text{——— (b)}$$

Now $|z| = r$ and $|\zeta - z|^2 = (e^{i\theta} - re^{i\phi})(e^{-i\theta} - re^{-i\phi}) = 1 - re^{i(\phi-\theta)} - re^{i(\theta-\phi)} + r^2$

$$= 1 - 2r\cos(\phi - \theta) + r^2$$

From (b):

$$\boxed{u(r,\phi) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(\theta)\, \frac{1-r^2}{1 - 2r\cos(\phi-\theta) + r^2}\, d\theta} \quad \text{——— (c)}$$

This is the (deservedly) famous <u>Poisson Integral Formula.</u> It gives the values of u inside the circle as an integral over the boundary values $u(\theta)$. The $v(r,\phi)$ is functionally identical.
This equation immediately shows how uniqueness is proved. If $u_1(\theta), u_2(\theta)$ are boundary values then $\epsilon(\theta) = u_1(\theta) - u_2(\theta) = 0$ (since $u_1 = u_2$ on the boundary) and $\epsilon(r,\phi) = 0$ inside (QED).

What happens when we examine the "+" sign case?

$$\frac{\zeta}{\zeta - z} + \frac{\zeta}{\zeta - 1/\bar{z}} = \frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{z} - \bar{\zeta}} = \frac{\zeta(\bar{\zeta} - \bar{z}) - \bar{z}(\zeta - z)}{(\zeta - z)(\bar{\zeta} - \bar{z})} = \frac{\zeta\bar{\zeta} - \zeta\bar{z} - \zeta\bar{z} + z\bar{z}}{|\zeta - z|^2}$$

$$= \frac{\zeta\bar{\zeta} + z\bar{z} - \zeta\bar{z} - z\bar{\zeta} + (z\bar{\zeta} - \zeta\bar{z})}{|\zeta - z|^2} = \frac{(\zeta - z)(\bar{\zeta} - \bar{z})}{|\zeta - z|^2} + \frac{(z\bar{\zeta} - \zeta\bar{z})}{|\zeta - z|^2} = 1 + \frac{z\bar{\zeta} - \overline{z\bar{\zeta}}}{|\zeta - z|^2}$$

$$= 1 + \frac{2i\,\mathrm{Im}(\bar{\zeta}z)}{|\zeta - z|^2}$$

Thus: $$2\pi f(z) = \int_0^{2\pi} f(\zeta)\, d\theta + 2i \int_0^{2\pi} \frac{f(\zeta)\,\mathrm{Im}(\bar{\zeta}z)}{|\zeta - z|^2}\, d\theta \quad \text{——— (d)}$$

Again: $\zeta = e^{i\theta}$, $z = re^{i\phi} \Rightarrow \bar{\zeta}z = re^{i(\phi-\theta)}$ and $|\zeta - z|^2 = 1 - 2r\cos(\phi-\theta) + r^2$ so that $\dfrac{\mathrm{Im}(\bar{\zeta}z)}{|\zeta - z|^2} = \dfrac{r\sin(\phi-\theta)}{1 - 2r\cos(\phi-\theta) + r^2}$. Substitution into (d) gives:

$$2\pi\left(u(r,\phi)+i v(r,\phi)\right)=\int_0^{2\pi}u(\theta)d\theta+i\int_0^{2\pi}v(\theta)d\theta+2i\int_0^{2\pi}u(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}d\theta$$
$$-2\int_0^{2\pi}v(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}d\theta$$

Hence: $$v(r,\phi)=\frac{1}{2\pi}\int_0^{2\pi}v(\theta)d\theta+\frac{1}{\pi}\int_0^{2\pi}u(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}d\theta$$

From the mean-value theorem $\frac{1}{2\pi}\int_0^{2\pi}v(\theta)d\theta=v(0)$ so that

$$v(r,\phi)=v(0)+\frac{1}{\pi}\int_0^{2\pi}u(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}d\theta \quad\text{———}\quad (e)$$

Once $u(\theta)$ is given, $v(r,\phi)$ is determined to within the arbitrary constant $v(0)$.

Combining (c) and (e) gives:

$$f(z)=u(r,\phi)+iv(r,\phi)=iv(0)+\frac{1}{2\pi}\int_0^{2\pi}u(\theta)\,\frac{1+i2r\sin(\phi-\theta)-r^2}{1-2r\cos(\phi-\theta)+r^2}\,d\theta \quad\text{——}\quad (f)$$

For a circle of radius R, replace r by r/R

Fourier Expansion of a Harmonic Function:

We write (f) as $f(z)=iv(0)+\frac{1}{2\pi}\int_0^{2\pi}u(\theta)\frac{\zeta+z}{\zeta-z}d\theta$ $\left(\begin{array}{l}\text{where } \zeta=e^{i\theta}\\ \text{and } z=re^{i\phi}\end{array}\right)$

Also $\frac{\zeta+z}{\zeta-z}=\frac{\zeta+z}{\zeta(1-z/\zeta)}=\frac{1}{1-z/\zeta}+\frac{z/\zeta}{1-z/\zeta}=\sum_{n=0}^{\infty}(z/\zeta)^n+\frac{z}{\zeta}\sum_{n=0}^{\infty}(z/\zeta)^n$ since $|z/\zeta|\le 1$.

$$=1+2\sum_{n=0}^{\infty}(z/\zeta)^n$$

Then $f(z)=iv(0)+\frac{1}{2\pi}\int_0^{2\pi}u(\theta)\left[1+2\sum_{n=0}^{\infty}(z/\zeta)^n\right]d\theta=iv(0)+\frac{1}{2\pi}\int_0^{2\pi}u(\theta)d\theta+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}u(\theta)r^n e^{i(\phi-\theta)n}d\theta$

$$=iv(0)+\frac{1}{2\pi}\int_0^{2\pi}u(\theta)d\theta+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}u(\theta)r^n\left[\cos n\phi\cos n\theta+\sin n\phi\sin n\theta+i\left(\sin n\phi\cos n\theta-\cos n\phi\sin n\theta\right)\right]d\theta$$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \left\{ \cos n\phi \int_0^{2\pi} u(\theta)\cos n\theta d\theta + \sin n\phi \int_0^{2\pi} u(\theta)\sin n\theta d\theta \right\}$$

$$+ i\left[v(0) + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \left\{ \sin n\phi \int_0^{2\pi} u(\theta)\cos n\theta d\theta - \cos n\phi \int_0^{2\pi} u(\theta)\sin n\theta d\theta \right\}\right.$$

Let $\alpha_n = \frac{1}{\pi}\int_0^{2\pi} u(\theta)\cos n\theta \, d\theta$ and $\beta_n = \frac{1}{\pi}\int_0^{2\pi} u(\theta)\sin n\theta \, d\theta$. Then

$$u(r,\phi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n \left\{ \alpha_n \cos n\phi + \beta_n \sin n\phi \right\}$$

$$v(r,\phi) = v(0) + \sum_{n=1}^{\infty} r^n \left\{ \alpha_n \sin n\phi - \beta_n \cos n\phi \right\}$$

These are Fourier series expansions of the harmonic functions $u(r,\phi)$, $v(r,\phi)$.

Note: In this section the mean-value property has been used, viz.,

$$f(z_0) = \frac{1}{2\pi r}\int_{\theta=0}^{2\pi} f(z_0 + re^{i\theta})d\theta \quad \text{with } f(z_0) \text{ analytic.}$$

Note: We can derive the Poisson Integral Formula directly from the solution of an elliptic boundary-value problem using separation of variables. We solve $\partial^2 T/\partial r^2 + (1/r)\partial T/\partial r + (1/r^2)\partial^2 T/\partial\theta^2 = 0$ in $0 \le r < R$, $0 \le \theta < 2\pi$ subject to $T(R,\theta) = f(\theta)$. Let $T = R(r)\Theta(\theta)$ to find the two ODEs $\Theta'' - \beta^2\Theta = 0$, $R'' + (1/r)R' - (\beta^2/r^2)R = 0$. The condition $\Theta(0) = \Theta(2\pi)$ gives $\beta_n = n^2$ $n = 0, 1, 2, \ldots$ and $\Theta_n = \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$ as the eigenfunctions in θ. For $R(r)$ we obtain the Cauchy-Euler equation $r^2 R'' + rR' - \beta^2 R = 0$, with $\beta = n$. For $n=0$, $R_0 = C_1 + C_2 \ln r$ and for $n>0$ $R_n = C_1 r^n + C_2 r^{-n}$. Bounded solutions are sought as $r \to 0$ so $R_n = r^n$, $n = 0, 1, 2, \ldots$.

Then

$$T(r,\theta) = \sum_{n=0}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta = a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \left[\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta\right].$$

At the boundary $r = R$

$$T(R,\theta) = f(\theta) = a_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta$$

So $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi$ and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi \, d\phi$, $b_n = \frac{1}{\pi}\int_0 f(\phi)\sin n\phi \, d\phi$, $n = 1, 2, \ldots$

Substitution gives $T(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \frac{1}{\pi}\int_0^{2\pi} f(\phi)[\cos n\theta\cos n\phi + \sin n\theta\sin n\phi]\,d\phi$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\phi)\right]d\phi$$

But $\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)\cos n(\theta-\phi) = Re\left[\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{in(\theta-\phi)}\right] = Re\left[\sum_{n=0}^{\infty}\left(\rho e^{i\chi}\right)^n\right]$ $\rho = r/R \le 1$, $\chi = \theta-\phi$

$$= Re\left[\sum_{n=1}^{\infty} z^n\right] = Re\left(\frac{z}{1-z}\right) = Re\left(\frac{\rho e^{i\chi}}{1-\rho e^{i\chi}}\right)$$

$$= Re\left(\frac{\rho e^{i\chi}(1-\rho e^{-i\chi})}{(1-\rho e^{i\chi})(1-\rho e^{-i\chi})}\right) = Re\left(\frac{\rho e^{i\chi}-\rho^2}{1-2\rho\cos\chi+\rho^2}\right)$$

$$= \frac{\rho\cos\chi-\rho^2}{1-2\rho\cos\chi+\rho^2}$$

Then $T(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1 + 2\left(\frac{\rho\cos\chi-\rho^2}{1-2\rho\cos\chi+\rho^2}\right)\right]d\phi = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\phi)\,d\phi}{r^2-2rR\cos(\theta-\phi)+R^2}$

Let $\varrho = r/R$ to obtain, finally

$$T(\varrho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,\frac{1-\varrho^2}{\varrho^2-2\varrho\cos(\theta-\phi)+1}\,d\phi$$

This is the same result derived previously. We note two things:

1) $T(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi$ = average of boundary values.

2) $\lim_{\varrho\to1^-} T(\varrho,\theta) = f(\theta)$, i.e., $\lim_{\varrho\to1^-}\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\phi)(1-\varrho^2)}{1-2\varrho\cos(\theta-\phi)+\varrho^2}\,d\phi = f(\theta)$.