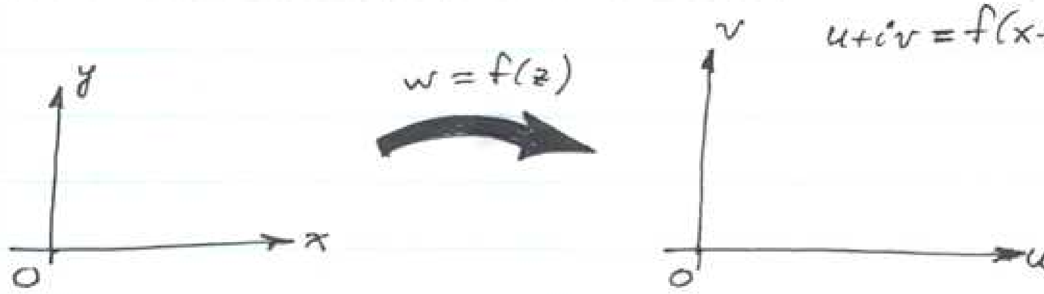
The Function as a Transformation of Coordinates: $w = f(z)$

$u + iv = f(x+iy)$



Example: $w = f(z) = z^2$

$$u + iv = (x+iy)^2 = (x^2 - y^2) + i(2xy)$$

$$\left.\begin{array}{l} u = x^2 - y^2 \\ v = 2xy \end{array}\right\} \text{transformation from } (x,y) \text{ to } (u,v) \text{ (vice-versa)}$$

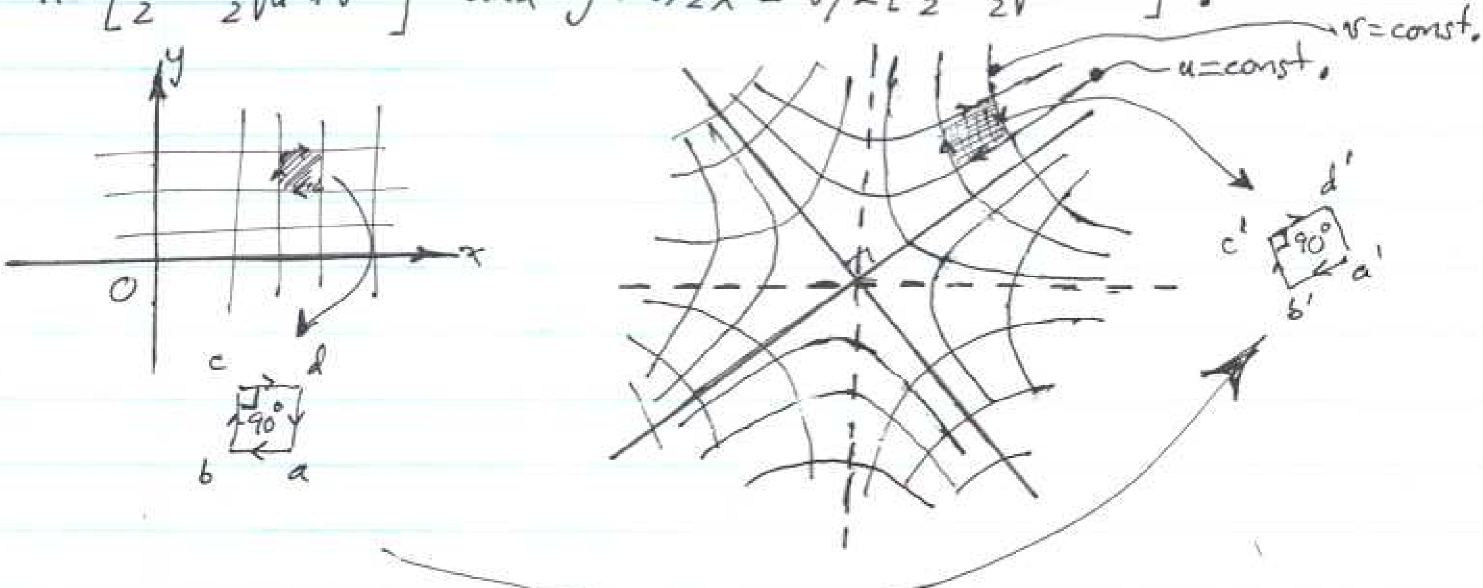
Note: $f(z) = z^2 : f'(z) = 2z : f''(z) = 2 : f'''(z) = \dots = f^{(n)}(z) = \dots = 0$

This function is analytic to all orders.

Write $u(x,y), v(x,y)$: $y = \frac{v}{2x} \Rightarrow u = x^2 - \frac{v^2}{4x^2} \Rightarrow x^4 - x^2 u - \frac{v^2}{4} = 0$

Then $x^2 = \frac{u}{2} \pm \frac{1}{2}\sqrt{u^2+v^2}$. Must choose "+" sign. Then

$x = \left[\frac{u}{2} + \frac{1}{2}\sqrt{u^2+v^2}\right]^{\frac{1}{2}}$ and $y = v/2x = v/2\left[\frac{u}{2} + \frac{1}{2}\sqrt{u^2+v^2}\right]^{\frac{1}{2}}$.



angles preserved, direction of traversal preserved.

Theorem: $f(z) = u(x,y) + i v(x,y)$

$\lim_{z \to z_0} f(z) = w_0$ iff $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Continuity does not always imply differentiability.
The existence of a derivative does imply continuity of the function.

Cauchy-Riemann (C-R) Equations:

$$f(z) = u(x,y) + i v(x,y)$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\begin{pmatrix} u(x_0+\Delta x, y_0+\Delta y) + \\ i v(x_0+\Delta x, y_0+\Delta y) \end{pmatrix} - \begin{pmatrix} u(x_0,y_0) + \\ i v(x_0,y_0) \end{pmatrix}}{\Delta x + i \Delta y}$$

$$= \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)}{\Delta x + i\Delta y} + \frac{i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right]}{\Delta x + i\Delta y}.$$

Let $\Delta z = \Delta x + i\cancel{\Delta y}^{\,0} \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Let $\Delta z = \cancel{\Delta x}^{\,0} + i\Delta y \Rightarrow f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

$\Rightarrow$ C-R: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Example: $f(z) = z^2 = (x^2 - y^2) + i(2xy) = u + iv$

$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 2x$ ✓

$\frac{\partial v}{\partial x} = 2y = -\frac{\partial}{\partial y}(x^2 - y^2) = 2y$ ✓

The C-R eq^n's hold true.

From the C-R equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$ and $-\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial x^2}$

Require functions $u(x,y)$, $v(x,y)$ s.t. $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$ so

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0 \end{aligned}\right\}$$ u, v Harmonic functions

Polar form: $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} z = re^{i\theta}$ $\quad w = f(z) = u(r,\theta) + i v(r,\theta)$

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{u(r_0+\Delta r, \theta_0+\Delta\theta) + i v(r_0+\Delta r, \theta_0+\Delta\theta) - \left[u(r_0,\theta_0) + i v(r_0,\theta_0)\right]}{(r_0+\Delta r)e^{i(\theta_0+\Delta\theta)} - r_0 e^{i\theta_0}}$$

leads to: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$; $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$

Exercise: Show that $f'(z) = \left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]e^{-i\theta} = \frac{1}{r}\left[\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right]e^{-i\theta}$, which leads to the polar C-R equations.

Note: From C-R $\Rightarrow$ $\left(\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y}\right)\cdot\left(\hat{x}\frac{\partial v}{\partial x} + \hat{y}\frac{\partial v}{\partial y}\right) = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$

Thus, $\underline{\nabla} u \cdot \underline{\nabla} v = 0$ so the surfaces of contant u, v are $\perp$.

Exercise: If $H_{xx} + H_{yy} = P(x,y)$ show that $w = f(z) = u + iv$, $z = x + iy$ leads to $H_{uu} + H_{vv} = \frac{P(x,y)}{|f'(z)|^2} \equiv \rho(u,v)$. Find the transformation for $P(x,y) = re^{-r}$, $r = \sqrt{x^2+y^2}$ and $f(z) = z^2$.

Recapitulate:

Analytic function: Function $f(z)$ is analytic at z_0 if $f'(z)$ exists at z_0 and in a "δ- neighborhood" of z_0

$f'(z)$ exists everywhere in this ball.

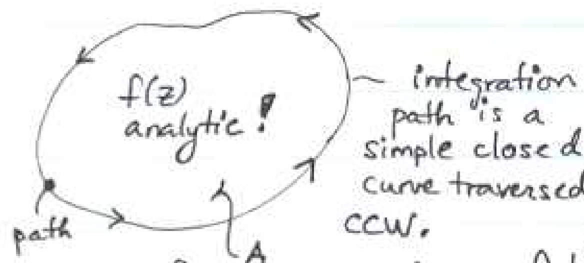
Example: $f(z) = \frac{1}{z}$; $f'(z) = -\frac{1}{z^2}$

Not analytic at $z = 0$ (analytic everywhere else).

A consequence of analyticity is the C-R equations so $f(z) = u + iv$ leads to $u_x = v_y$, $u_y = -v_x$. There are many consequences of this deceptively simple result.

Analytic Functions in Complex Analysis: There are several important consequences that ensue when function $f(z)$ is analytic.

Cauchy Formula: $\oint f(z)\,dz = 0$



Proof: Use Green's theorem

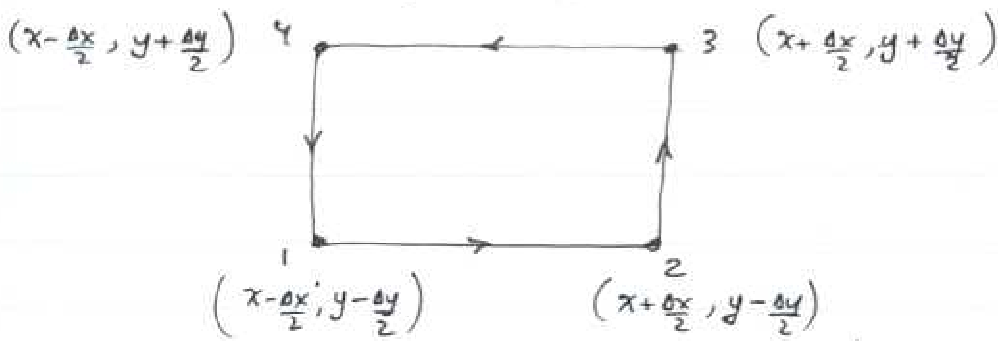
$$\oint [P\,dx + Q\,dy] = \iint_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dy\,dx$$ where $P(x,y)$, $Q(x,y)$ are $\text{fct's}^n(x,y)$

Then $$\oint f(z)\,dz = \oint (u+iv)(dx+i\,dy) = \oint [\underset{P}{u}\,dx + \underset{Q}{\underbrace{(-v)}}\,dy] + i\oint [\underset{P}{v}\,dx + \underset{Q}{u}\,dy]$$

$$= \iint_A \underbrace{\left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]}_{=0} dy\,dx + i\iint_A \underbrace{\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]}_{=0} dy\,dx = 0 \quad \text{by C-R.}$$

QED.

Exercise: Using the path shown prove Green's theorem for continuous, differentiable functions $P(x,y)$, $Q(x,y)$.



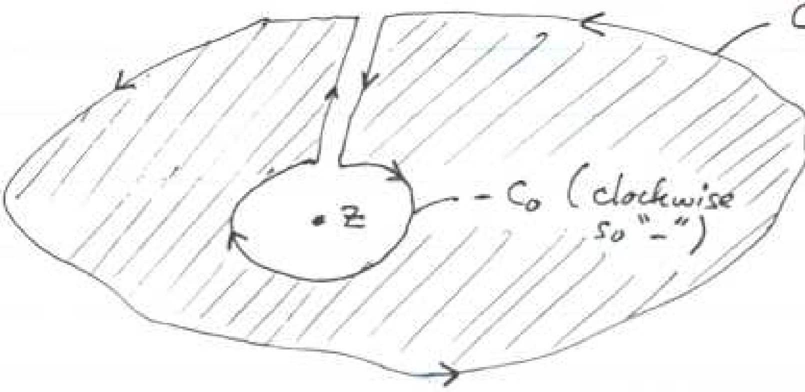
Thus, show that $$\oint [P\,dx + Q\,dy] = \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 = \iint_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dy\,dx.$$

One of the most spectacular results of all complex analysis is the Cauchy integral formula, which is just a simple identity but opens doors to all of complex functional analysis:

Cauchy Integral Formula: $$f(z) = \frac{1}{2\pi i}\oint \frac{f(\xi)}{\xi - z}\,d\xi$$

$f(z)$ analytic but $f(\xi)/(\xi - z)$ not analytic at $\xi = z$

To prove this result let's examine the vicinity of point z. We remember that $f(z)$ is analytic everywhere in this vicinity:



In the shaded region bounded by $-C_0 + C$ $\displaystyle\int_{-C_0+C} \frac{f(\xi)}{\xi - z}\,d\xi = 0$ since $f(\xi)/(\xi - z)$ is analytic there. The shaded region excludes z.

Then

$$\oint_{-C_0} \frac{f(\xi)}{\xi - z}\,d\xi + \oint_{C} \frac{f(\xi)}{\xi - z}\,d\xi = 0 \;\Rightarrow\; \boxed{\oint_{C} \frac{f(\xi)}{\xi - z}\,d\xi = \oint_{+C_0} \frac{f(\xi)}{\xi - z}\,d\xi}$$

clockwise!

counterclockwise!

<u>Simplicity:</u> Choose C_0 as unit circle $R = 1$, $(\xi - z = e^{i\theta})$ let $f(z) = \text{constant} \equiv K$.

$$\oint_{C_0} \frac{f(\xi)}{\xi - z}\,d\xi = \int_{\theta=0}^{2\pi} \frac{K}{e^{i\theta}}\,d(e^{i\theta}) = \int_0^{2\pi} \frac{K i e^{i\theta}}{e^{i\theta}}\,d\theta = 2\pi i K.$$

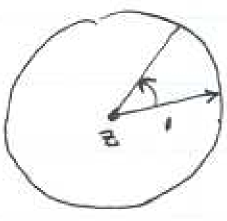
Thus $\displaystyle K = \frac{1}{2\pi i} \int_{C_0} \frac{K}{\xi - z}\,d\xi$

Generally:

$$\oint_{C_0} \frac{f(\xi)}{\xi - z}\,d\xi = \oint \frac{f(\xi) + f(z) - f(z)}{\xi - z}\,d\xi = f(z) \oint_{C_0} \frac{d\xi}{\xi - z} + \oint_{C_0} \frac{f(\xi) - f(z)}{\xi - z}\,d\xi$$

$\xi - z = 1 e^{i\theta}$

$d\xi = i\,d\theta\, e^{i\theta}$



$$= f(z)\cdot 2\pi i + \int_{\theta=0}^{2\pi} \frac{f(z + e^{i\theta}) - f(z)}{\cancel{e^{i\theta}}}\, i \cancel{e^{i\theta}}\,d\theta$$

$$= 2\pi i\, f(z) + i \underbrace{\int_{\theta=0}^{2\pi} [f(z + e^{i\theta}) - f(z)]\,d\theta}_{0} = 2\pi i f(z).$$

<u>Exercise:</u> Show that $\displaystyle\int_{\theta=0}^{2\pi} [f(z + e^{i\theta}) - f(z)]\,d\theta = 0$ for any analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$. You need only demonstrate it for the j^{th} term.