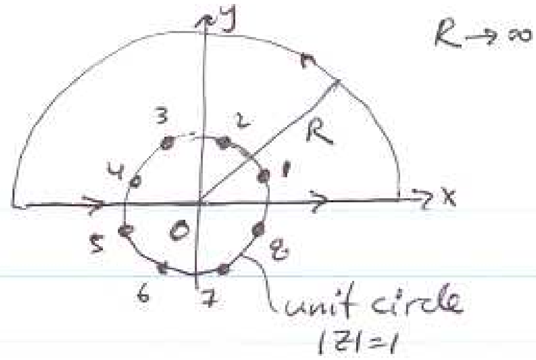
Examples of Simple Contour Integration

In the simple examples that follow, the only challenge is to locate the poles, calculate the residues, and evaluate their sums after having determined the relation between the complex contour integral and the original (presumably real) integral.

Common tricks:

1. If an integral contains $\sin x, \cos x$ in the integrand numerator, replace by e^{ix} and take Im or Re of resulting complex integral.
2. With polynomials in the denominator, find always the zeros
3. In evaluating sums of residues for functions such as $\frac{1}{1+z^n}$ one will obtain residues like $\delta + \delta^m + \delta^{2m} + \delta^{3m} + \dots$. Series like this can be summed exactly. For example let $S_9 = \delta + \delta^3 + \delta^5 + \delta^7 + \delta^9$, which we rewrite as $S_9 = \delta(1+\delta^2+\delta^4+\delta^6+\delta^8)$. Now let $S_9 + \delta^{11} = \delta(1+\delta^2+\delta^4+\delta^6+\delta^8+\delta^{10}) = \delta[1+\delta^2(1+\delta^2+\delta^4+\delta^6+\delta^8)] = \delta[1+\delta S_9]$. In other words the sum $S_9 + \delta^{11}$ can also be written as $\delta + \delta^2 S_9$. We equate these expressions to find $S_9 = \delta(1-\delta^{10})/(1-\delta^2)$. This procedure is easy to generalize for other progressions (even powers, all powers, etc.).

(1) Evaluate $I = \int_{-\infty}^{\infty} \frac{x^4}{1+x^8}\,dx$.



$$I = \int_{-\infty}^{\infty} \frac{x^4}{1+x^8}\,dx = \oint_C \frac{z^4}{1+z^8}\,dz \quad \text{on } C$$

$$= \int_{-\infty}^{\infty} \frac{x^4}{1+x^8}\,dx + \underbrace{\int_{\theta=0}^{\pi} \frac{R^4 e^{4i\theta}}{1+R^8 e^{8i\theta}}\, iRe^{i\theta}\,d\theta}_{\text{as } R\to\infty \text{ this contribution vanishes.}}$$

Poles of $(1+z^8)^{-1}$:

$1+z^8=0 \Rightarrow z^8 = -1 = e^{i\pi}$

Inside C: $z_1 = e^{i\pi/8}$
$z_2 = e^{3i\pi/8}$
$z_3 = e^{5i\pi/8}$
$z_4 = e^{7i\pi/8}$

(z_5, z_6, z_7, z_8 are outside C)

Thus, $I = \oint_C \frac{z^4}{1+z^8}\,dz = 2\pi i \sum_{j=1}^{4}(\text{residues inside C})_j$

Can use partial fractions but it is easier to write $\frac{P(z)}{Q(z)} = \frac{z^4}{1+z^8}$ and $(\text{residue})_i = \frac{P(z_i)}{Q'(z_i)} = \frac{z_i^4}{8z_i^7} = \frac{1}{8z_i^3}$

$$\Rightarrow \quad I = \frac{2\pi i}{8}\left(\frac{1}{z_1^3} + \frac{1}{z_2^3} + \frac{1}{z_3^3} + \frac{1}{z_4^3}\right) = \frac{\pi i}{4}\left(e^{-3i\pi/8} + e^{-9i\pi/8} + e^{-15i\pi/8} + e^{-21i\pi/8}\right).$$

$$= \frac{\pi i}{4} S \; ; \quad S = \delta + \delta^3 + \delta^5 + \delta^7 \quad \text{where} \quad \delta = e^{-3i\pi/8}$$

Evaluate the sum $S(\delta)$:
$$S = \delta\left[1+\delta^2+\delta^4+\delta^6\right]$$
$$= \delta\left[1+\delta^2(1+\delta^2+\delta^4+\delta^6-\delta^6)\right]$$
$$= \delta\left[1+\delta\cdot\delta(1+\delta^2+\delta^4+\delta^6) - \delta^8\right]$$
$$\Rightarrow S = \delta\left[1+\delta S - \delta^8\right]$$
$$\Rightarrow S = \frac{\delta(1-\delta^8)}{1-\delta^2} = \frac{\cancel{\delta}\left(\frac{1}{\delta^4}-\delta^4\right)\delta^4}{\cancel{\delta}\left(\frac{1}{\delta}-\delta\right)}$$

But $\delta^4 = \left(e^{-3i\pi/8}\right)^4 = e^{-\frac{12i\pi}{8}} = e^{-\frac{3i\pi}{2}} = \cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2} = i \Rightarrow \delta^4\left(\frac{1}{\delta^4}-\delta^4\right) = i\left(\frac{1}{i}-i\right) = 2$

And $\frac{1}{\delta} - \delta = e^{3i\pi/8} - e^{-3i\pi/8} = \cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8} - \left(\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right) = 2i\sin\frac{3\pi}{8}$.

Hence
$$I = \frac{\pi i}{4} S = \frac{\pi i}{4}\,\frac{i\left(\frac{1}{i}-i\right)}{2i\sin 3\pi/8} = \frac{\pi i}{4}\,\frac{(1+1)}{2i\sin\frac{3\pi}{8}} = \frac{\pi}{4\sin 3\pi/8}$$

(2) Evaluate $I = \int_{-\infty}^{\infty} \frac{x\sin x}{1+x^4}\,dx$.

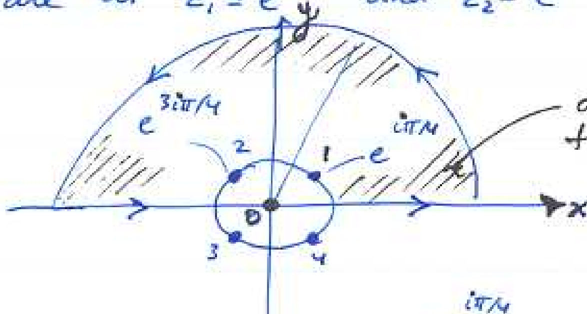
$$I = \int_{-\infty}^{\infty} \frac{x\sin x}{1+x^4}\,dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^4}\,dx\right)$$

Since $$\oint \frac{z e^{iz}}{1+z^4}\,dz = \int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^4}\,dx + \underbrace{\int_{\theta=0}^{\pi} \frac{R\cdot e^{i\theta}\, e^{iRe^{i\theta}}\, iRe^{i\theta}}{1+R^4 e^{4i\theta}}\,d\theta}_{\text{vanishes as } R\to\infty}$$

Then $$I = \operatorname{Im}\oint \frac{z e^{iz}}{1+z^4}\,dz = \operatorname{Im}\left(2\pi i\cdot \sum_{j=1}^{M}(\text{residue})_j\right)$$

<u>Residues</u>: $\frac{P(z)}{Q'(z)} = \frac{z e^{iz}}{4z^3} = \frac{e^{iz}}{4z^2}$

Poles of $(1+z^4)^{-1}$ have $(1+z^4)=0$ or $z^4=-1=e^{i\pi}$. Inside the contour C the poles are at $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$:



domain of integration lies always to the left of the traversal path.

$$\sum_{j=1}^{2}(\text{residue})_j = \frac{e^{iz_1}}{4z_1^2} + \frac{e^{iz_2}}{4z_2^2} = \frac{e^{ie^{i\pi/4}}}{4(e^{i\pi/4})^2} + \frac{e^{ie^{3i\pi/4}}}{4(e^{3i\pi/4})^2} = \frac{1}{2} e^{-\frac{\sqrt{2}}{2}} \sin\frac{\sqrt{2}}{2}.$$

Then $$I = \operatorname{Im}\left(2\pi i\cdot \frac{1}{2} e^{-\frac{\sqrt{2}}{2}} \sin\frac{\sqrt{2}}{2}\right) = \pi e^{-\frac{\sqrt{2}}{2}} \sin\frac{\sqrt{2}}{2}.$$

③ Evaluate $I = \int_{-\infty}^{\infty} \frac{\cos x}{(x+\alpha)^2+\beta^2}\,dx$.

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{(x+\alpha)^2+\beta^2}\,dx = \mathrm{Re}\left\{\int_{-\infty}^{\infty} \frac{e^{ix}}{(x+\alpha)^2+\beta^2}\,dx\right\} = \mathrm{Re}\left\{\oint_C \frac{e^{iz}}{(z+\alpha)^2+\beta^2}\,dz\right\}$$

$$= \mathrm{Re}\left[2\pi i \sum(\text{residues})\right].$$

Zeros of $(z+\alpha)^2+\beta^2 = z^2+2\alpha z+(\alpha^2+\beta^2)$:

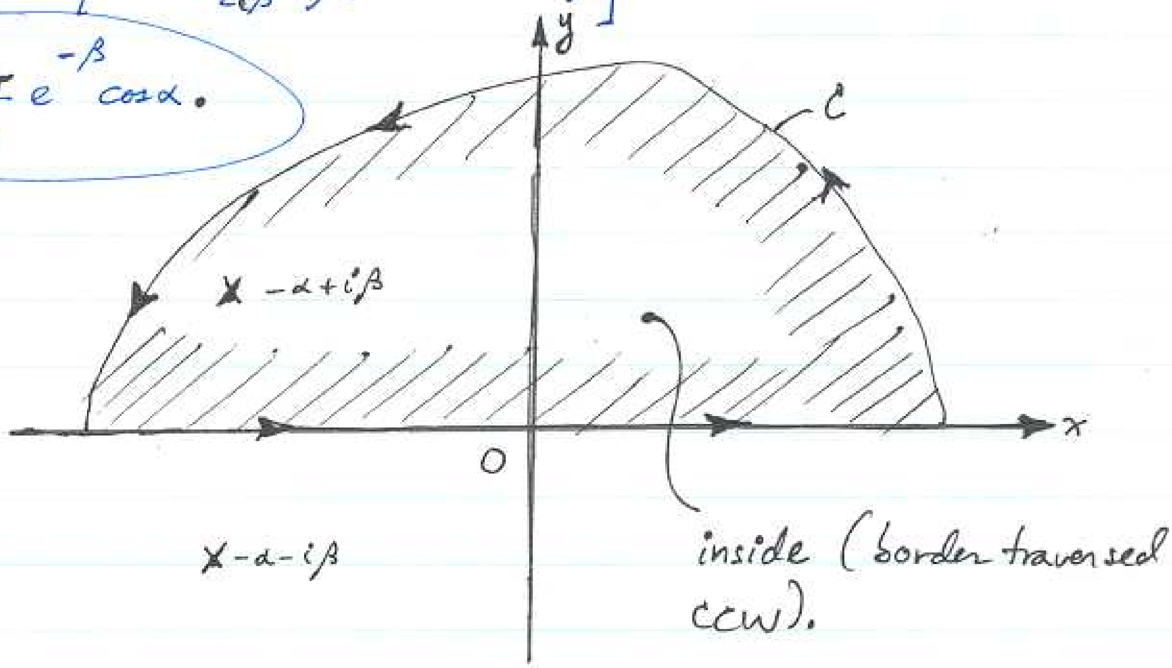
Quadratic formula: $z_{1,2} = -\alpha \pm \frac{1}{2}\sqrt{4\alpha^2-4(\alpha^2+\beta^2)} = -\alpha \pm i\beta$.

We complete the contour in the upper half plane so the only residue occurs at $z=-\alpha+i\beta$ above the real axis. The residue there is

$$\frac{P(z)}{Q'(z)} = \frac{e^{iz}}{2(z+\alpha)}\bigg|_{z=-\alpha+i\beta} = \frac{e^{i(-\alpha+i\beta)}}{2(\cancel{-\alpha}+i\beta+\cancel{\alpha})} = \frac{e^{-\beta-i\alpha}}{2i\beta} = \frac{e^{-\beta}}{2i\beta}(\cos\alpha - i\sin\alpha).$$

Thus $I = \mathrm{Re}\left[2\pi i\left(\frac{e^{-\beta}}{2i\beta}\right)(\cos\alpha - i\sin\alpha)\right]$

$\Rightarrow$ $I = \frac{\pi}{\beta} e^{-\beta}\cos\alpha.$



Example: $I(a) = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \qquad |a| < 1.$

Let $z = e^{i\theta}$, $dz = i\,d\theta\, z$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}(2\cos\theta)$ ✓

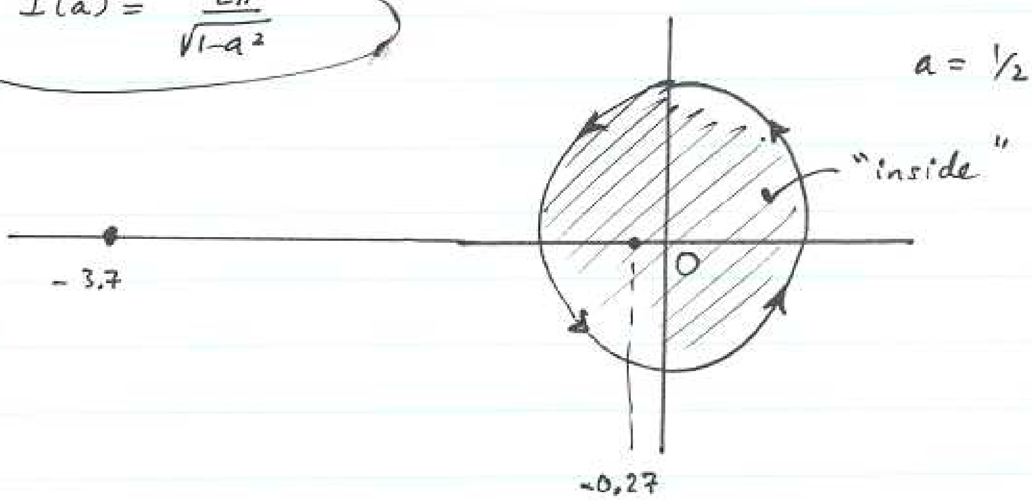
Then
$$I = \frac{1}{i} \int_{\substack{|z|=1 \\ \text{(unit circle)}}} \frac{1}{z} \frac{dz}{1 + \frac{a}{2}\left(z + \frac{1}{z}\right)} = \frac{2}{i} \int_{\substack{|z|=1 \\ \text{(unit circle)}}} \frac{dz}{a\left(z^2 + \frac{2}{a}z + 1\right)}$$

Zeros of $z^2 + \frac{2z}{a} + 1$ are $z_{1,2} = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}$. The residue with the "$-$" sign does not lie inside the unit circle since $\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1} > 1$. The residue with the "$+$" sign does lie inside the unit circle. Thus,

$$I = \frac{2}{i} \int_{\substack{\text{unit} \\ \text{circle}}} \frac{dz}{a\underbrace{(z - z_1)}_{\text{inside}}\underbrace{(z - z_2)}_{\text{outside}}} = \frac{2}{i} \cdot 2\pi i \cdot (\text{residue at } z = z_1)$$

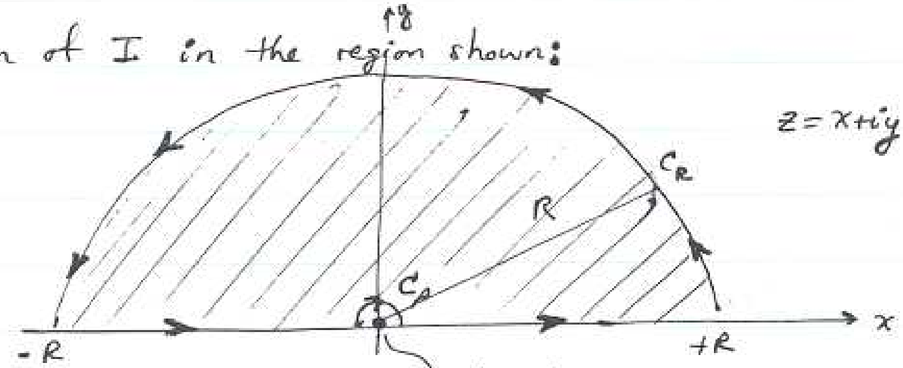
$$= \frac{2}{i} \cdot 2\pi i \left(\frac{1}{a(z_1 - z_2)}\right) = \frac{4\pi}{a \cdot 2\sqrt{\frac{1}{a^2} - 1}} = \frac{2\pi}{\sqrt{1 - a^2}}$$

Thus: $I(a) = \frac{2\pi}{\sqrt{1 - a^2}}$



If the path were traversed CW then the "outside would be the inside" and the residue at $z = z_2$ would be used to evaluate the integral.

Example: $I = \int_0^\infty \frac{\sin \alpha x}{x}\,dx$, $\alpha > 0$. Has a singularity at $x = 0$.
Since I is also expressible as $I = \int_{-\infty}^{0} \frac{\sin \alpha x}{x}\,dx$ then $2I = \int_{-\infty}^{\infty} \frac{\sin \alpha x}{x}\,dx = \mathrm{Im}\left\{\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\,dx\right\}$. I.e., $I = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\,dx$. Lets evaluate the complex continuation of I in the region shown:



$-\rho$ $\quad$ ρ $\quad$ pole at $z=0$.

$$\oint_C \frac{e^{i\alpha z}}{z}\,dz = 0 = \int_{-R}^{-\rho} \frac{e^{i\alpha x}}{x}\,dx + \int_{\rho}^{R} \frac{e^{i\alpha x}}{x}\,dx + \int_{C_\rho} \frac{e^{i\alpha z}}{z}\,dz + \int_{C_R} \frac{e^{i\alpha z}}{z}\,dz$$

the contour C **excludes** the singularity (pole) at the origin.

traversed clockwise from $\theta = \pi$ to $\theta = 0$

$$\int_{\theta=0}^{\pi} \frac{e^{i\alpha R\cos\theta - \alpha R\sin\theta}}{Re^{i\theta}} \cdot iRe^{i\theta}\,d\theta = 0 \quad (\text{since } \sin\theta > 0 \text{ when } 0 < \theta < \pi)$$

On C_ρ: Let $z = \rho e^{i\theta}$; $0 < \theta < \pi$: $\int_{C_\rho} \frac{e^{i\alpha z}}{z}\,dz = \int_{\theta=\pi}^{\theta=0} \frac{e^{i\alpha(\rho\cos\theta + i\rho\sin\theta)}}{\rho e^{i\theta}}\,\rho i e^{i\theta}\,d\theta$

Take the limit as $\rho \to 0$ to get $\int_{C_\rho} \frac{e^{i\alpha z}}{z}\,dz = \int_{\pi}^{0} i\,d\theta = -i\pi$.

Also, as $\rho \to 0$ and $R \to \infty$ we get

$$0 = \int_{-\infty}^{0^-} \frac{e^{i\alpha x}}{x}\,dx + \int_{0^+}^{\infty} \frac{e^{i\alpha x}}{x}\,dx + \lim_{\rho\to 0}\int_{C_\rho} \frac{e^{i\alpha z}}{z}\,dz$$

$$0 = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\,dx + (-i\pi).$$

Thus: $\frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\,dx = \frac{1}{2}\,\mathrm{Im}(+i\pi) = \frac{\pi}{2}$.

$$\int_0^\infty \frac{\sin \alpha x}{x}\,dx = \frac{\pi}{2}, \quad \alpha > 0$$

Until now, poles of the integrands have always been first order. Clearly, poles can have a higher order. By the Cauchy theorem we can write $f(z) = (1/2\pi i)\int_C [f(\xi)/(\xi - z)]d\xi$. This integral contains a pole of order one. The function $g(z) = f'(z) = (1/2\pi i)\int[f(\xi)/(\xi - z)^2]d\xi$ contains a pole of order two. Similarly, f'', f''', etc. contain poles of order three, four, etc. In short, it is not difficult to construct integrals with poles of all orders.

Let's write, from the Cauchy formula:

$$\frac{d^n f(z)}{dz^n} \equiv f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

We suppose that there is a pole of order n in the integrand $f(\xi)$ near $\xi = z$. Thus, we write $f(\xi) = g(\xi)/(\xi - z)^n$, with $g(\xi)$ analytic near $\xi = z$. By the Cauchy theorem, we have

$$\frac{d^{n-1} g(z)}{dz^{n-1}} = \frac{(n-1)!}{2\pi i}\int_C \frac{g(\xi)}{(\xi - z)^n}\, d\xi = \frac{(n-1)!}{2\pi i}\int_C f(\xi)\, d\xi .$$

Since the residue of a function is defined as $\frac{1}{2\pi i}\int_C f(\xi)\,d\xi$ we have, for function $f(z)$:

$$\text{residue} = \frac{1}{2\pi i}\int_C f(\xi)\,d\xi = \frac{1}{(n-1)!}\,\frac{d^{n-1} g(z)}{dz^{n-1}} = \frac{1}{(n-1)!}\lim_{\xi \to z}\frac{d^{n-1}}{d\xi^{n-1}}\left[(\xi - z)^n f(\xi)\right]$$

<u>Example:</u> $f(\xi) = \xi^m/(\xi - z)^n$ with $m = 1$, $n = 2$. Then residue $= \frac{1}{(1-1)!}\,\frac{dz}{dz} = 1$. Thus $\int_C \frac{\xi}{(\xi - z)^2}\,d\xi = 2\pi i$ (contour C encloses z).