Contour Integration:

Use the residue theorem to evaluate complex integrals over closed paths, obtaining results thereby for real integrals evaluated over real paths. Recalling

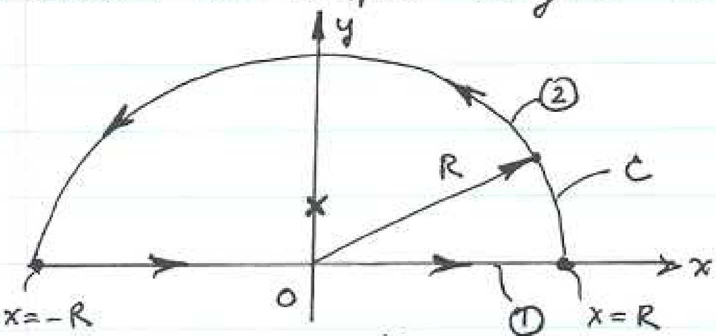
$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} (\text{residues})_j$$

where $\sum_j^n (\text{residues})_j$ is the sum of the n residues at all the poles and singularities of $f(z)$ inside C. For a simple pole the residue is $C_{-1} = \lim_{z\to z_0} (z-z_0)f(z)$. For $f(z) = \phi(z)/\chi(z)$ where $\chi(z) = (z-z_0)\chi'(z_0) + \frac{(z-z_0)^2}{2!}\chi''(z_0) + \dots$ one finds $C_{-1} = \phi(z_0)/\chi'(z_0)$. If z_0 is a pole of order m the residue is

$$C_{-1} = \frac{1}{(m-1)!} \lim_{z\to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) \right]$$

Example: $I = \int_0^\infty \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_0^R \frac{dx}{1+x^2} = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Let us evaluate the complex integral $J = \frac{1}{2} \oint \frac{dz}{1+z^2}$ along the path shown:



$$J = \frac{1}{2}\oint \frac{dz}{1+z^2} = \frac{1}{2}\int_{①} \frac{dz}{1+z^2} + \frac{1}{2}\int_{②} \frac{dz}{1+z^2}$$

Along ①: $z = x + i\overset{0}{y} = x$, $dz = dx$ from $x = -R$ to $x = +R$

Along ②: $z = Re^{i\theta}$, $dz = iRe^{i\theta}\,d\theta$ from $\theta = 0$ to $\theta = 2\pi$

$$J = \frac{1}{2}\int_{-R}^{R} \frac{dx}{1+x^2} + \frac{1}{2}\int_{\theta=0}^{2\pi} \frac{iRe^{i\theta}\,d\theta}{1+R^2e^{2i\theta}}$$

Let $R\to\infty$ so that $J = \frac{1}{2}\oint \frac{dz}{1+z^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = I$. Thus, we can evaluate I by evaluating J. We have $J = \frac{1}{2}\oint \frac{dz}{1+z^2} = \frac{1}{2}\oint \frac{dz}{(z+i)(z-i)} =$

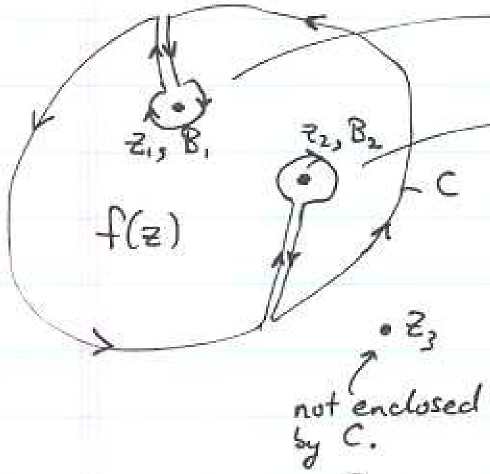
$\frac{1}{2}\oint \left(\frac{i/2}{z+i} - \frac{i/2}{z-i} \right) dz = \frac{1}{2} \cdot 2\pi i\,(-i/2) = \frac{\pi}{2}$.

residue is outside C

residue is inside C, at $z = i$

Thus, $I = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

Reiteration: If the pole or singularity is not enclosed by the contour its contribution is zero.



contributes $2\pi i B_1$ (nothing from z_2 which is external to this part).

contributes $2\pi i B_2$ (nothing from z_1).

$$\oint_C f(z)dz = 2\pi i B_1 + 2\pi i B_2 = 2\pi i (B_1 + B_2)$$

<u>Example:</u> $\quad I = \int_0^{\pi} \ln(\sin x)dx$

Lets evaluate the integral $J \equiv \oint_{\substack{\text{unit}\\ \text{circle}\\ |z|=1}} \ln\left(\frac{\epsilon}{i} + \sin\theta\right)d\theta$ and let $\epsilon \to 0$ at end.

How is I related to J?

$$Re(J) = Re \oint_{|z|=1} \ln(-i\epsilon + \sin\theta)d\theta$$

As $\epsilon \to 0$ $\quad Re(J) = \int_0^{2\pi} \ln|\sin\theta|d\theta = 2\int_0^{\pi} \ln|\sin\theta|d\theta = 2I$

$\left(\begin{array}{l} \text{since } -i\epsilon + \sin\theta \\ = \sqrt{\sin^2\theta + \epsilon^2}\, e^{-i\tan^{-1}\epsilon/\sin\theta} \\ \text{and as } \epsilon \to 0 \text{ get} \\ \sqrt{\sin^2\theta} = |\sin\theta|. \end{array}\right)$

Thus, $\quad I = \frac{1}{2} Re(J)$

and we evaluate I by calculating J and then finding its real part. On the unit circle $z = 1 \cdot e^{i\theta}$

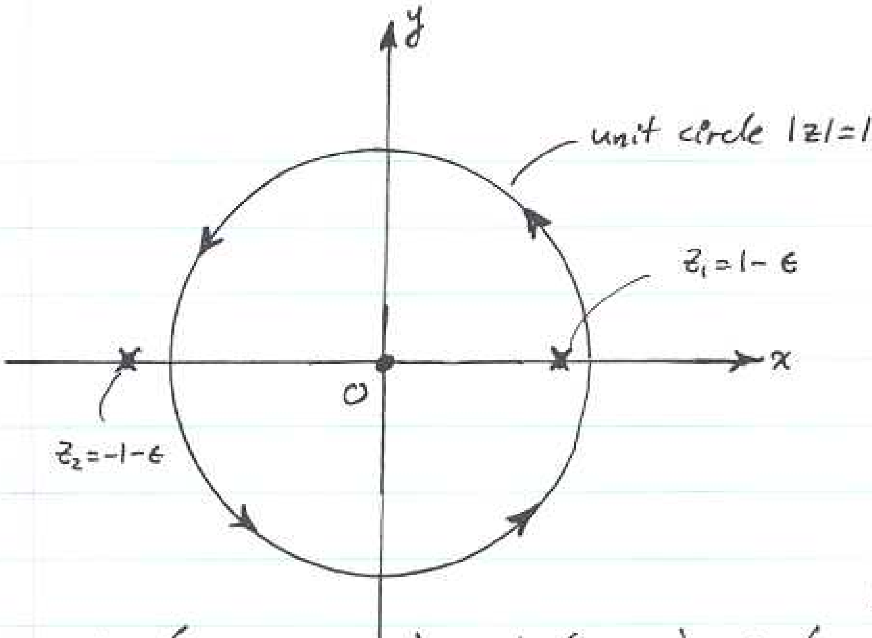
$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta$$

$$\Rightarrow \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow \frac{\epsilon}{i} + \sin\theta = \frac{\epsilon}{i} + \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2 + 2\epsilon z - 1}{2iz} = \frac{(z-z_1)(z-z_2)}{2iz}$$

where $z_1 = -\epsilon + \sqrt{1+\epsilon^2}$ and $z_2 = -\epsilon - \sqrt{1+\epsilon^2}$. As $\epsilon \to 0$ get $z_1 \doteq 1 - \epsilon + \dots$ and $z_2 = -(1+\epsilon) + \dots$ so z_1 is inside $|z| = 1$ and z_2 is outside $|z| = 1$.

See the figure on next page... ↗



analytic inside the unit circle.

Thus: $\ln\left(\frac{(z-z_1)(z-z_2)}{2iz}\right) = \ln\left(\frac{z-z_1}{z}\right) + \ln\left(\frac{z-z_2}{2i}\right)$

analytic outside the unit circle

Write $z = e^{i\theta}$, $dz = i d\theta e^{i\theta} = i d\theta z$ so $d\theta = dz/iz$ on the unit circle.

Then:

$$J = \oint_{\substack{\text{unit}\\ \text{circle}\\ |z|=1}} \ln\left(\frac{\epsilon}{i} + \sin\theta\right) d\theta = \oint_{|z|=1} \ln\left(\frac{(z-z_1)(z-z_2)}{2iz}\right)\frac{dz}{iz}$$

$$= \oint_{|z|=1} \frac{\ln((z-z_1)/z)}{iz} dz + \oint_{|z|=1} \frac{\ln((z-z_2)/2i)}{iz} dz$$

We put a negative sign in front of the first integral and reverse the contour so $\oint_{|z|=1} \frac{\ln((z-z_1)/z)}{iz} dz = -\oint_{|z|=1} \frac{\ln((z-z_1)/z)}{iz} dz$. The function is analytic outside the circle but there are no poles there (the pole $z=0$ is inside the circle) so the residue is zero. The second integral is essentially $f(z) = \phi(z)/\chi(z)$, $\phi(z) = \ln((z-z_2)/2i)$ (analytic inside circle), $\chi(z) = iz$ so the residue is $C_{-1} = \frac{1}{i}\ln\left(\frac{-z_2}{2i}\right) = \frac{1}{i}\ln\left(\frac{\epsilon+\sqrt{1+\epsilon^2}}{2i}\right)$. Thus,

$$J = 2\pi i\left(\frac{1}{i}\ln\left(\frac{\epsilon+\sqrt{1+\epsilon^2}}{2i}\right)\right) = 2\pi\ln\left(\frac{\epsilon+\sqrt{1+\epsilon^2}}{2i}\right) = 2\pi\left[\ln(\epsilon+\sqrt{1+\epsilon^2}) - \ln 2 - \frac{i\pi}{2}\right]$$

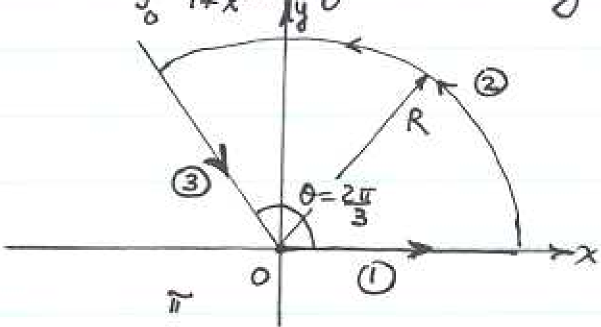
$$Re(J) = 2\pi \ln\left(\epsilon + \sqrt{1+\epsilon^2}\right) - 2\pi \ln 2$$

As $\epsilon \to 0$ get $\quad Re(J) = -2\pi \ln 2$

Then $\quad I = \frac{1}{2} Re(J)_{\epsilon=0} = -\pi \ln 2$

It is worthwhile studying this outstanding exercise (taken from Carrier, Krook and Pearson, Functions of a Complex Variable, McGraw-Hill, 1966).

<u>Exercise:</u> Evaluate $I = \int_0^\infty \frac{dx}{1+x^3}$ by calculating $J = \oint \frac{dz}{1+z^3}$ along the contour shown:
(Mathews & Walker pp. 68-69)



on ①: $0 \le x < \infty$

on ②: $0 \le \theta \le \frac{2\pi}{3}$; $R \to \infty$

on ③: $z = re^{i\theta}$ with $\theta = \frac{2\pi}{3}$ and $0 \le r \le R$ $(R \to \infty)$.

<u>Exercise:</u> Evaluate $I = \int_0^\pi \frac{d\theta}{a+b\cos\theta}$ $\quad a>b>0$ noting that the integrand is even and then integrating along the unit circle with $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ and $\cos\theta = (e^{i\theta}+e^{-i\theta})/2 = (z+1/z)/2$ (Mathews & Walker, pp. 69-70).

<u>Exercise:</u> Evaluate $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1} dx$, $0<a<1$ using $J = \oint \frac{e^{az}}{e^z+1} dz$ along a suitably-chosen contour (Mathews & Walker, pp. 71-72).