Branching & Branched Mappings:

Example: $w = f(z) = \frac{1}{\sqrt{z}} = \frac{1}{\sqrt{re^{i\theta}}} = r^{-1/2} e^{-i\theta/2} \equiv \rho e^{i\phi}$.

$$\rho = r^{-1/2} \; ; \; \phi = -\theta/2 .$$

Start at $\theta = -\pi$ and traverse θ by $\pi/2$ increments to $+\pi$.

$\underline{\theta = -\pi :}$ $z = re^{-i\pi} = -r$
$w = z^{-1/2} = r^{-1/2} e^{i\pi/2} = i\rho.$

$\underline{\theta = -\pi/2 :}$ $z = re^{-i\pi/2} = -ir$
$w = z^{-1/2} = r^{-1/2} e^{i\pi/4} = \rho \frac{\sqrt{2}}{2}(1+i).$

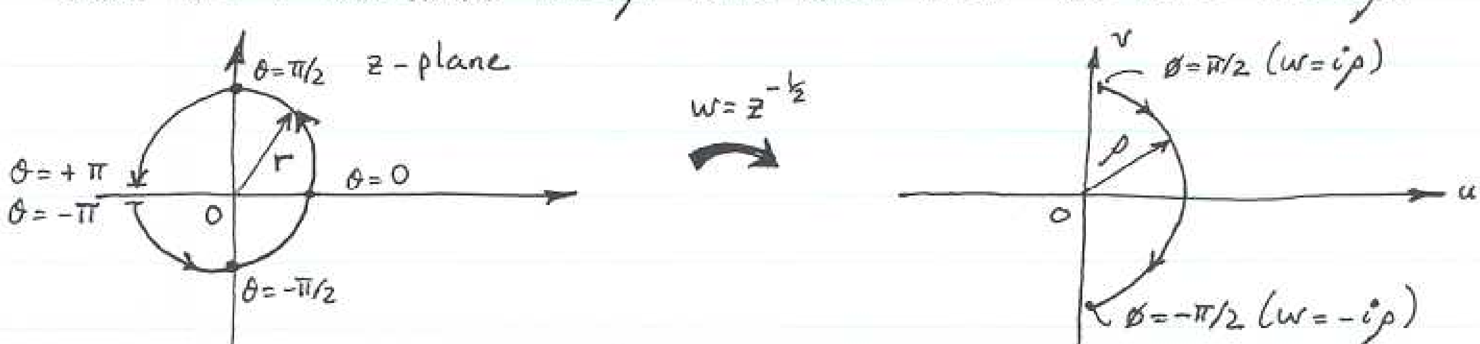
$\underline{\theta = 0 :}$ $z = r$
$w = z^{-1/2} = r^{-1/2} = \rho.$

$\underline{\theta = \pi/2 :}$ $z = ir$
$w = r^{-1/2} e^{-i\pi/2} = \rho \frac{\sqrt{2}}{2}(1-i)$

$\underline{\theta = \pi :}$ $z = re^{i\pi} = -r$
$w = z^{-1/2} = r^{-1/2} e^{-i\pi/2} = -i\rho.$

θ	ϕ	$w = 1/\sqrt{z}$
$-\pi$	$\pi/2$	$i\rho$
$-\pi/2$	$\pi/4$	$\rho\frac{\sqrt{2}}{2}(1+i)$
0	0	ρ
$\pi/2$	$-\pi/4$	$\rho\frac{\sqrt{2}}{2}(1-i)$
π	$-\pi/2$	$-i\rho$

So, at $\theta = -\pi$ and $\theta = +\pi$ we have $z = -r$ (same value) but when $\theta = -\pi$ we have $w = i\rho$ and when $\theta = \pi$ we have $w = -i\rho$.



Let $\theta = -\pi + \epsilon$, $\theta = \pi - \epsilon$ and consider $\epsilon \to 0$:

$$w = r^{-1/2} e^{-i\theta/2} = \frac{1}{\sqrt{r}} \begin{Bmatrix} e^{-\frac{i}{2}(\pi-\epsilon)} \\ e^{-\frac{i}{2}(-\pi+\epsilon)} \end{Bmatrix} = \frac{1}{\sqrt{r}} \begin{Bmatrix} -ie^{i\epsilon/2} \\ ie^{-i\epsilon/2} \end{Bmatrix}.$$

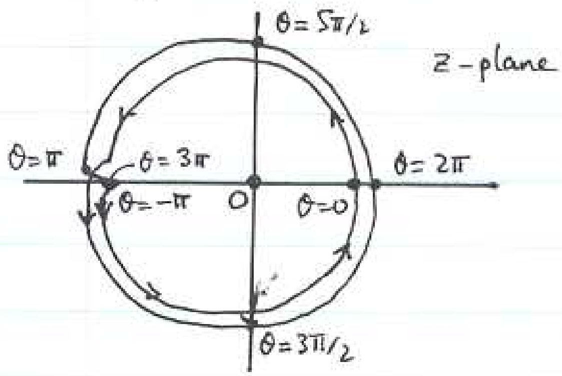
Clearly $-ie^{i\epsilon/2} \neq ie^{-i\epsilon/2}$ as $\epsilon \to 0$ since $-i \neq i$.

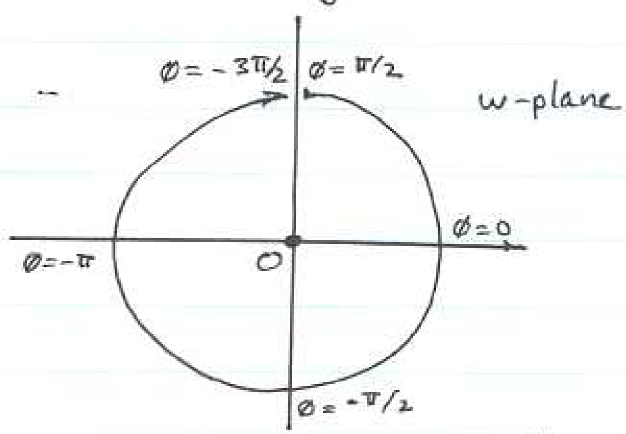
The problem with this mapping in the z-plane is its multivaluedness. Clearly, an entire circle in the z-plane maps only to the right-half semicircle in the w-plane.

Lets continue to vary θ in the $+$ (CW) direction

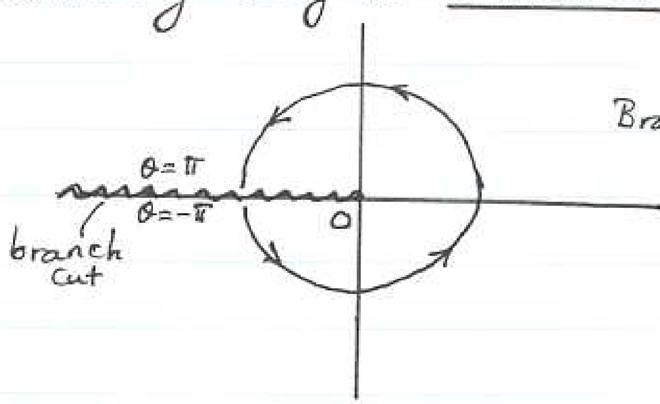
θ	π	$3\pi/2$	2π	$5\pi/2$	3π	$7\pi/2$	....
ϕ	$-\pi/2$	$-3\pi/4$	$-\pi$	$-5\pi/4$	$-3\pi/2$	$-7\pi/4$	....
$w = 1/\sqrt{z}$	$-i\rho$	$\frac{\sqrt{2}}{2}(1-i)\rho$	$-\rho$	$-\frac{\sqrt{2}}{2}(1+i)\rho$	$i\rho$	$\frac{\sqrt{2}}{2}(1+i)\rho$	....

At $\theta = 3\pi$ we return to the value $z^{-1/2} = i\rho$ that we had at $\theta = -\pi$. Thus, two circuits in the z-plane correspond with only one circuit in the w-plane:





In the z-plane we say the function $f(z) = z^{-1/2}$ possesses two "levels." On the first level, $-\pi < \theta < \pi$. Then a "stairwell" places us on a second level (or stack) [same radius $1/\sqrt{r}$] with $\pi < \theta < 3\pi$. A second stairwell connects this function to the $\theta = -\pi$ values and the first level circuit repeats itself. We represent this z-plane function "discontinuity" using a <u>branch cut</u>:



Branch cut for $z^{-1/2}$.

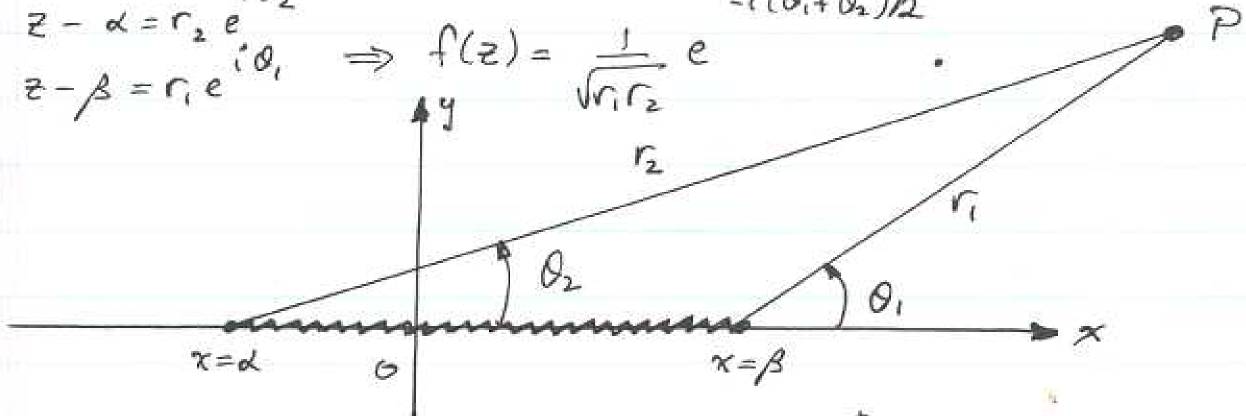
In the region exclusive of the branch cut function $w = 1/\sqrt{z}$ is single-valued. The branch cut connects $z=0$ and the point at ∞.

Exercise: Carry out the preceding analysis for $w = f(z) = z^{\frac{1}{3}}$. How many "levels" does function $f(z) = z^{1/n}$ have?

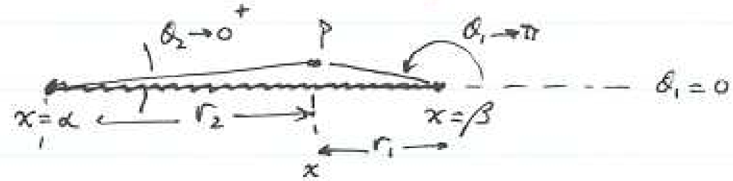
Sometimes functions have multiple singular points, which are connected by a branch cut:

Example: $f(z) = \dfrac{1}{\sqrt{z-\alpha}\sqrt{z-\beta}}$. Singular at $z=\alpha$ and $z=\beta$.

Let $z - \alpha = r_2 e^{i\theta_2}$, $z - \beta = r_1 e^{i\theta_1}$ $\Rightarrow f(z) = \dfrac{1}{\sqrt{r_1 r_2}} e^{-i(\theta_1+\theta_2)/2}$.



Top side of cut: $\theta_2 = 0$, $\theta_1 = \pi$

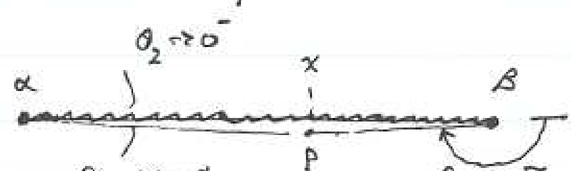


$r_2 = x - \alpha$ $\quad \theta_2 = 0$
$r_1 = \beta - x$ $\quad \theta_1 = \pi$

$$\Rightarrow f(z)\Big|_{\substack{\text{top}\\\text{side}}} = \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} e^{-i(\pi+0)/2} = \frac{-i}{\sqrt{(x-\alpha)(\beta-x)}},$$

with $\alpha < x < \beta$ on the top side of the cut.

Bottom side of cut: $\theta_2 = 0$, $\theta_1 = -\pi$



$r_2 = x - \alpha$
$r_1 = \beta - x$
$\theta_2 = 0$
$\theta_1 = -\pi$

$$\Rightarrow f(z)\Big|_{\substack{\text{bottom}\\ \text{side}}} = \frac{e^{-i(-\pi+0)/2}}{\sqrt{(x-\alpha)(\beta-x)}} = \frac{i}{\sqrt{(x-\alpha)(\beta-x)}}.$$

We see clearly that on the top side $f(z) = -i g(z)$ whereas on the bottom side $f(z) = i g(z)$, where $g(z) = \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}$. Function $f(z)$ takes separate and distinct values on either side of the cut but outside of that region $f(z)$ is single-valued in the z-plane.

<u>Exercise:</u> For the preceding example calculate the values of $f(z)$ very near to the branch cut at $\theta_1 = \pi + \epsilon$ and $\theta_1 = -\pi + \epsilon$ and show that the function values are not equal.

<u>Exercise:</u> Allow point P to rotate about $x = \beta$ (starting with $\theta_1 = -\pi$). Determine the number of "levels" for this branched function. Explain or draw these levels, indicating transitions to different levels.